Jon's Big Beautiful Book of Maths Formulas

Second edition

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Contents

Preface

This is a comprehensive collection of the bewildering array of maths formulae and maths jargon that are commonly encountered in high school maths, and a little beyond. It is aimed at those who are less confident in maths and those who just need a quick reference. It uses standard notation. Units mainly follow the SI standard.

Many people who use maths become sloppy in the use of notation. The aim of notation is not to confuse or hide the meaning from the uninitiated, like some sort of secret code, rather it is to ensure that maths is written clearly, succinctly, elegantly and most importantly, free of ambiguity. One of the aims of this work, thus, is to demonstrate how the jargon and notation should be used and to highlight types of notation that are easily confused. Maths can be confusing, so the glossary describes concepts as plainly as possible, using examples where appropriate.

There is no index, as it is assumed that readers will be able to search the PDF. Likewise, terms are sometimes not defined in the glossary if they are clearly defined elsewhere. When in doubt, search.

In addition, there are many many good online resources for maths: Wolfram Alpha (`www.wolframalpha.com/`), Khan Academy (`www.khanacademy.org/`), Wikipedia (of course; `en.wikipedia.org/wiki/Mathematics`, and many other pages) and GeoGebra (`www.geogebra.org/`), to mention a few.

This was inspired by my favourite equation (1), and made possible by the beauty of LaTeX.

For any corrections, amendments or suggestions, please email me at `jonathan.webley@gmail.com`.

1 Notation

Variable	Use
a, b, c, k	Often used for constants
e	Euler's number, $2.718\,281\,828\,459\,0$
f, g, h	Often used for functions
h	Height of a triangle, cylinder, etc.
i, j	Often used to indicate a term in a list
i	Square root of -1, $\sqrt{-1}$
l	Length, line
n	Often used to indicate the number of items in a list
r	Radius of a circle, sphere, cylinder, etc.
t	Time
x	General unknown, polynomials, quadratic equations, x-axis
y	General unknown, y-axis
z	z-axis in 3D, complex number
A	Area
O	The origin of a coordinate system, $(0, 0)$
V	Volume
Lower-case Greek letters	
δ (delta)	Used for a small difference or small change
ϵ (epsilon)	Often used for a very small number
θ (theta)	An angle
μ (mu)	Mean or average
π	Pi, $3.141\,592\,653\,589\,8$; $\pi \approx \dfrac{22}{7} = 3.142\,86$
σ (sigma)	Standard deviation
Upper-case Greek letters	
Δ (delta)	Used for a difference or a change, e.g. ΔE could be the change in energy over a time Δt

Symbol	Meaning	Example
$=$	equals	$1 = 1$
$\equiv$	equivalent to, identical to	
$\neq$	not equals	$1 \neq 2$
$\approx$ or $\simeq$	approximately equals	$0.1111 \approx 0.1112$
$\sim$	of the order of; sampled from a statistical distribution	
$\cong$	congruent to	
$<$	less than	$1 < 2$
$\not<$	not less than	$7 \not< 5$
$\leq$ or $\leqslant$	less than or equals	$1 \leq 2 \leq 2$
$>$	greater than	$3 > 2$
$\not>$	not greater than	$2 \not> 5$
$\geq$ or $\geqslant$	greater than or equals	$3 \geq 2 \geq 2$
$\propto$	proportional to	$y \propto x$ if $y = kx$
$+$	addition or plus	$1 + 1 = 2$
$-$	subtraction or minus	$3 - 1 = 2$
$\pm$	plus or minus, meaning either treat as a plus or treat as minus	4 ± 1 means $4 + 1$ or $4 - 1$ (or both)
$\mp$	minus or plus, i.e., use the operators in the opposite order to $\pm$	
$\times$	multiplication or times	$3 \times 2 = 6$
$\div$ or $/$	division	$6 \div 2 = 6/2 = 3$
!	*factorial*	$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
$\sqrt{\ }$	square root	$\sqrt{16} = 4$
$\sqrt[n]{\ }$	nth root	$\sqrt[4]{16} = 2$
$*$	various uses in maths, so should always be defined; commonly used in programming for multiplication but this usage is wrong in maths since it can be misleading	
$\circ$	used for some binary operator	$f \circ g$
$^\circ$	degrees (angle or temperature)	1°, 1°C
$\lvert x \rvert$	modulus or absolute value of $x \in \mathbb{R}$: $\begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$	
∞	*infinity*	
$\sum$	sum (capital sigma)	$\sum_{i=1}^{i=4} i = 1 + 2 + 3 + 4 = 10$
$\prod$	product (capital pi)	$\prod_{i=1}^{i=4} i = 1 \times 2 \times 3 \times 4 = 24$

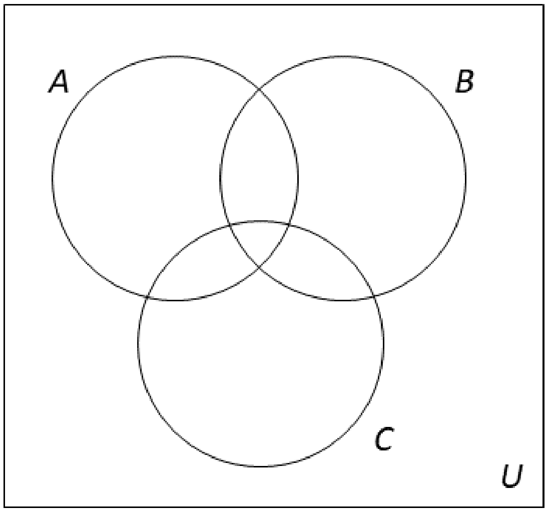
Symbol	Meaning	Example
∂	partial, used for partial derivatives	$\dfrac{\partial y}{\partial x}$
$\int$	integration (read as ‘integral of’)	
$\therefore$	therefore	
$\square$, $\blacksquare$	often used at the end of a proof	
$\parallel$	*parallel* to	
$\perp$	*perpendicular* to	
$\forall$	for all	$\forall i$ (for all i)
$\exists$	there exists	$\exists i$ (there exists an i)
$\Rightarrow$	implies	$x = y \Rightarrow y = x$
$\to$	asymptotically tends to	$\lim_{n\to\infty} 1/n = 0$
$\mapsto$	maps to	if $y = f(x) = hx^2$ then $f : x \mapsto x^2$

1.1 Notation used in programming

Note that, in general, this notation should *not* be used when writing maths. Each programming language uses its own set of notation, so the examples are typical forms of the syntax used.

<table>
<tr><th>Symbol</th><th>Meaning</th><th>Example</th><th></th></tr>
<tr><td>++</td><td>add 1</td><td>add 1 to a</td><td>a++</td></tr>
<tr><td>--</td><td>subtract 1</td><td>subtract 1 from a</td><td>a--</td></tr>
<tr><td>+=</td><td>add</td><td>add 5 to a</td><td>a+=5</td></tr>
<tr><td>-=</td><td>subtract</td><td>subtract 5 from a</td><td>a-=5</td></tr>
<tr><td>*=</td><td>multiply</td><td>multiply a by 5</td><td>a*=5</td></tr>
<tr><td>/=</td><td>divide</td><td>divide a by 5</td><td>a/=5</td></tr>
<tr><td>*</td><td>multiplication</td><td>2×3</td><td>2*3</td></tr>
<tr><td>^</td><td>exponent</td><td>2^3</td><td>2^3</td></tr>
<tr><td>:=</td><td>assignment</td><td>set a equal to 1</td><td>a:=1</td></tr>
<tr><td>==</td><td>is equal to (in a condition)</td><td>if $a = 1$ then...</td><td>if a==1 then...</td></tr>
<tr><td><></td><td>is not equal to (in a condition)</td><td>if $a \neq 1$ then...</td><td>if a<>1 then...</td></tr>
<tr><td>!=</td><td>is not equal to (in a condition)</td><td>if $a \neq 1$ then...</td><td>if a!=1 then...</td></tr>
<tr><td>&&</td><td>logical and (in a condition)</td><td>if $a = 1$ and $b = 2$ then...</td><td>if a==1 && b==2 then...</td></tr>
<tr><td>||</td><td>logical or (in a condition)</td><td>if $a = 1$ or $b = 2$ then...</td><td>if a==1 || b==2 then...</td></tr>
<tr><td>E</td><td>power of 10</td><td>2.123×10^5</td><td>2.123E05</td></tr>
</table>

2 Sets



Venn diagram

$A = \{a_1, a_2, a_3, \dots\}$	Set A has members or elements $a_1, a_2, a_3, \dots$
$\{x \mid \dots\}$	The set of all x such that $\dots$
$\emptyset$ or $\{\}$	the empty set, a set with no members
U	Universal set
$a \in A$	a is in A
$a \notin A$	a is not in A
$n(A)$	No of members in A
$A \cup B = B \cup A$	union, either in A OR in B or both
$A \cap B = B \cap A$	intersection, both in A AND in B
A'	not in A, complement of A
$A \backslash B$	in A but not in B
$A \subset B \supset A$	A is a proper subset of B
$A \subseteq B \supseteq A$	$A = B$ or A is a subset of B

Commutative

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associative

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Union is distributive over intersection:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Intersection is distributive over union:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$$

de Morgan's laws

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

Intervals

$$(a, b) =]a, b[= \{x \in \mathbb{R} \mid a < x < b\}$$
$$(a, b] =]a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$
$$[a, b) = [a, b[= \{x \in \mathbb{R} \mid a \leq x < b\}$$
$$[a, b] = [a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Cartesian product

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

3 Arithmetic

3.1 Types of number

Set	Symbol	Description	Definition
Naturals	$\mathbb{N}$	Counting numbers, positive integers. May or may not include 0.	$\{0, 1, 2, 3, \dots\}$ or $\{1, 2, 3, \dots\}$
Integers	$\mathbb{Z}$	Whole numbers	$\{0, \pm 1, \pm 2, \pm 3, \dots\}$
Positive integers	$\mathbb{Z}^+$	Whole numbers greater than 0	$\{1, 2, 3, \dots\}$
Negative integers	$\mathbb{Z}^-$	Whole numbers less than 0	$\{\dots, -3, -2, -1\}$
Rationals	$\mathbb{Q}$	Numbers that can be written as fractions, like $-1/2$, 0 and $3/1$	$\{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$
Irrationals		Numbers that cannot be written as a fraction, such as surds	
Transcendentals		Real or complex numbers that are not roots of a polynomial with integer coefficients, such as π and e	
Reals	$\mathbb{R}$	Rationals plus irrationals	
Positive reals	$\mathbb{R}^+, \mathbb{R}_{>0}$	Real numbers greater than 0	$\{x \mid x \in \mathbb{R}, x > 0\}$
Complex numbers	$\mathbb{C}$	Numbers involving $\mathrm{i} = \sqrt{-1}$	$\{a + \mathrm{i}\, b \mid a, b \in \mathbb{R}\}$

Countably infinite: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ $\quad$ $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Uncountably infinite: $\mathbb{R}, \mathbb{C}$

Order of calculation

BEDMAS	brackets, exponents, division/multiplication, addition/subtraction
BIDMAS	brackets, indices, division/multiplication, addition/subtraction
BODMAS	brackets, order, division/multiplication, addition/subtraction
	brackets, of/division/multiplication, addition/subtraction
PEMDAS	parentheses, exponents, multiplication/division, addition/subtraction

3.2 Properties of arithmetic operators

Commutative: $a \circ b = b \circ a$

Operator	Applies	Example or counterexample
Addition	Y	$1 + 2 = 3 = 2 + 1$
Subtraction	N	$1 - 2 = -1 \neq 1 = 2 - 1$
Multiplication	Y	$1 \times 2 = 2 = 2 \times 1$
Division	N	$1 \div 2 = 0.5 \neq 2 = 2 \div 1$

Associative: $a \circ (b \circ c) = (a \circ b) \circ c$

Operator	Applies	Example or counterexample
Addition	Y	$2 + (3 + 4) = 9 = (2 + 3) + 4$
Subtraction	N	$2 - (3 - 4) = 3 \neq -5 = (2 - 3) - 4$
Multiplication	Y	$2 \times (3 \times 4) = 24 = (2 \times 3) \times 4$
Division	N	$2 \div (3 \div 4) = 8/3 \neq 1/6 = (2 \div 3) \div 4$

Here, $\circ$ is any operator.

Multiplication is distributive over addition:

$$a \times (b + c) = (a \times b) + (a \times c)$$
$$2 \times (3 + 4) = 14 = (2 \times 3) + (2 \times 4)$$

Addition is *not* distributive over multiplication:

$$a + (b \times c) \neq (a + b) \times (a + c)$$
$$2 + (3 \times 4) = 24 \neq 30 = (2 + 3) \times (2 + 4)$$

3.3 Infinity and zero

The following are all for $k > 0$.

Addition:

$$\infty + k = \infty \qquad 0 + k = k$$
$$\infty + \infty = \infty \qquad 0 + 0 = 0$$
$$\infty + 0 = \infty$$

Subtraction:

$$\infty - k = \infty \qquad 0 - k = -k$$
$$k - \infty = -\infty \qquad k - 0 = k$$
$$\infty - \infty = \text{undefined} \qquad 0 - 0 = 0$$
$$\infty - 0 = \infty \qquad 0 - \infty = -\infty$$

Multiplication:

$$\infty \times k = \infty \qquad 0 \times k = 0$$
$$\infty \times -k = -\infty \qquad 0 \times -k = 0$$
$$\infty \times \infty = \infty \qquad 0 \times 0 = 0$$
$$\infty \times 0 = \text{undefined}$$

Division:

$$\frac{\infty}{k} = \infty \qquad \frac{0}{k} = 0$$
$$\frac{\infty}{-k} = -\infty \qquad \frac{0}{-k} = 0$$

$$\frac{k}{\infty} = 0 \qquad \frac{k}{0} = \infty$$
$$\frac{-k}{\infty} = 0 \qquad \frac{-k}{0} = -\infty$$

$$\frac{\infty}{\infty} = \text{undefined} \qquad \frac{0}{0} = \text{undefined}$$
$$\frac{\infty}{0} = \infty \qquad \frac{0}{\infty} = 0$$

Exponents:

$$k^{\infty} = \begin{cases} 0, & 0 < k < 1 \\ \infty, & k > 1 \end{cases}$$
$$(-k)^{\infty} = \begin{cases} 0, & 0 < k < 1 \\ -\infty, & k > 1 \end{cases}$$

$$k^0 = (-k)^0 = 1 \qquad -(k^0) = -1$$
$$\infty^k = \infty \qquad 0^k = 0$$
$$\infty^{-k} = \frac{1}{\infty^k} = 0 \qquad 0^{-k} = \frac{1}{0^k} = \infty$$
$$\infty^{\infty} = \infty \qquad 0^0 = \text{undefined}$$
$$\infty^0 = \text{undefined} \qquad 0^{\infty} = 0$$
$$1^{\infty} = \text{undefined} \qquad 1^0 = 1$$

3.4 Roman numerals

I = 1	C = 100
V = 5	D = 500
X = 10	M = 1000
L = 50	

Multiples of the same symbol are additive:

II = 2   III = 3

MM = 2000   MMM = 3000

A smaller number after a larger is additive:

VI = 6   XI = 11

LX = 60   MC = 1100

A smaller number before a larger is subtractive:

IV = 4   IX = 9

XL = 40   CM = 900

To give:

I = 1	XI = 11
II = 2	XII = 12
III = 3	XIII = 13
IV = 4	XIV = 14
V = 5	XV = 15
VI = 6	XVI = 16
VII = 7	XVII = 17
VIII = 8	XVIII = 18
IX = 9	XIX = 19
X = 10	XX = 20

MCMXCIX = 1999	MMI = 2001
MM = 2000	MMXX = 2020

4 Complex numbers

Cartesian form:

$$z = x + iy \text{ where } x, y \in \mathbb{R} \text{ and } z \in \mathbb{C}$$

Real part of z: $\mathrm{Re} = x$

Imaginary part of z: $\mathrm{Im} = y$

Polar form:

$$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\,\mathrm{cis}\,\theta = r\,\mathrm{e}^{\mathrm{i}\,\theta}$$

Modulus:

$$r = |z| = \sqrt{x^2 + y^2} \qquad (r \geq 0)$$

$$|z_1 z_2| = |z_1||z_2|, \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Argument:

$$\theta = \arg(z) = \tan^{-1}\frac{y}{x} \quad (\text{rad}, -\pi < \theta \leq \pi)$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2),$$
$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

For 2nd quadrant, $\theta = \pi + \tan^{-1}\frac{y}{x}$ (tan is -ive here).

For 3rd quadrant, $\theta = \tan^{-1}\frac{y}{x} - \pi$ (tan is +ive here).

$$x = r\cos\theta, \quad y = r\sin\theta$$

Complex conjugate:

$$\bar{z} = z^* = x - \mathrm{i}\,y$$
$$zz^* = x^2 + y^2 \in \mathbb{R}$$

Euler's formula:

$$\mathrm{e}^{\mathrm{i}\,\theta} = \cos\theta + \mathrm{i}\sin\theta$$

Euler's identity (for $\theta = \pi$):

$$\mathrm{e}^{\mathrm{i}\,\pi} = -1 \tag{1}$$

de Moivre:

$$z^n = r^n\,\mathrm{cis}(n\theta) = r^n(\cos n\theta + \mathrm{i}\sin n\theta) = r^n\,\mathrm{e}^{\mathrm{i}\,n\theta}$$

Roots of unity:

$$\{\mathrm{e}^{2k\pi\,\mathrm{i}\,/n} \mid k \in \{1, 2, \ldots, n\}\}$$

In an Argand diagram, multiplying by a power of i is equivalent to a rotation by an angle:

By	Rotation by
i	90°
$\mathrm{i}^2 = -1$	180°
$\mathrm{i}^3 = -\mathrm{i}$	270°
$\mathrm{i}^4 = 1$	360°

5 Sequences and series

A series is the sum of a sequence.

Linear series: First differences are constant
Quadratic series: Second differences are constant
Cubic series: Third differences are constant

$$\sum_1^n i = \frac{1}{2}n(n+1)$$
$$\sum_1^n i^2 = \frac{1}{6}n(n+1)(2n+1)$$

5.1 Arithmetic or linear series

nth term:

$$u_n = u_1 + (n-1)d$$

for first term u_1 and common difference d.

Sum of n terms:

$$\begin{aligned} S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ &= \frac{n}{2}(u_1 + u_n) \\ S_\infty &= \pm\infty \end{aligned}$$

$$n = \frac{u_n - u_1}{d} + 1$$

5.2 Geometric series

nth term:

$$u_n = u_1 r^{n-1}$$

for first term u_1 and common ratio r.

Sum of n terms:

$$S_n = \frac{u_1(r^n - 1)}{r-1} = \frac{u_1(1-r^n)}{1-r} = S_\infty(1-r^n)$$

Diverges if $|r| > 1$.

Sum of infinite terms:

$$S_\infty = \frac{u_1}{1-r}, \quad -1 < r < 1$$

5.3 Maclaurin series

Centred at the origin:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

Functions:

$$\begin{aligned} \mathrm{e}^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots, \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} \\ \tan x &= x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \frac{272x^3}{7!} + \dots \\ \arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \end{aligned}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots, \quad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots, \quad \text{for } -1 < x \leq 1, p \in \mathbb{R}$$

(cf. *Binomial theorem*)

for all real numbers x.

5.4 Taylor series

Centred at $x = a$:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

Approximation:

$$f_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x)$$

Lagrange form of the error term:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{where } c \in [a, x]$$

for some c, $0 < c < x$.

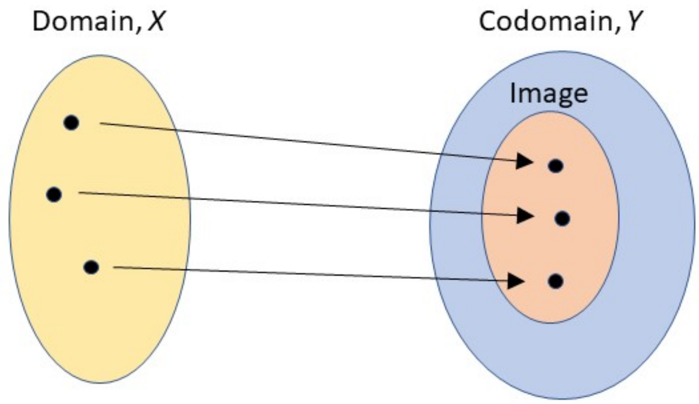
6 Functions

- A **relation** is a set of ordered pairs (x, y) that specifies corresponding values of the independent (x) and dependent (y) variables.
- A **function** is a relation between two sets in which every element of the first set (the input or independent variable) is mapped onto one and only one element of the second set (the output or dependent variable).

- An *input* is an object that is put into the function.

- The *domain* is the set of all inputs for which a function is defined.
- An *output* or *image* is the object that comes out of the function.
- The *range* is the set of actual output values of a function.
- The *codomain* is the set of all possible output values of a function.

The domain X is mapped to the range or image Y, which is a subset of the codomain.



$$f : X \mapsto Y$$

Even function:

$$f(-x) = f(x), \quad \text{e.g., } y = x^2, y = x^4$$

Odd function:

$$f(-x) = -f(x), \quad \text{e.g., } y = x^3, y = x^5$$

Surjective or onto: range = codomain

Injective: one-to-one

Bijective: Both of the above. Bijections have inverses

6.1 Inverses

If $y = f(x)$ then $x = f^{-1}(y)$.

$$f^{-1} \circ f(x) = f \circ f^{-1}(x) = x.$$

The inverse function is equivalent to a reflection of $y = f(x)$ in line $y = x$.

6.2 Composite functions or function of a function

$$h(x) = g(f(x)) = gf(x) = g \circ f(x)$$

In general, $gf(x) \neq fg(x)$.

Example

For $g(x)$ and $f(x)$, these are equivalent:

- $f(x) = x^2$ and $g(x) = 2x - 1$
- $f(x) = x^2$ and $g(y) = 2y - 1$, so $y = f(x) = x^2$
- $f : x \mapsto x^2$ and $g : x \mapsto 2x - 1$

Then

$$g \circ f(x) = g(f(x)) = g(x^2) = 2(x^2) - 1$$
$$f \circ g(x) = f(g(x)) = f(2x - 1) = (2x - 1)^2$$

6.3 Translations

	$a < 0$	$a = 0$	$a > 0$	Vector
Horizontal $f(x+a)$ $(x,y) \mapsto (x-a,y)$	shifted by a to the right	no change	shifted by a to the left	$\begin{pmatrix} -a \\ 0 \end{pmatrix}$
Vertical $f(x)+a$ $(x,y) \mapsto (x,y+a)$	shifted by a downwards	no change	shifted by a upwards	$\begin{pmatrix} 0 \\ a \end{pmatrix}$

6.4 Scaling

	$\lvert k \rvert < 1$	$k = 1$	$\lvert k \rvert > 1$	Vector
Vertical stretch $kf(x)$ $(x,y) \mapsto (x,ky)$	squeezed in y direction by k	no change	stretched in y direction by k	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
Horizontal stretch $f(x/k)$ $(x,y) \mapsto (kx,y)$	squeezed in x direction by k	no change	stretched in x direction by k	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$

6.5 General

For $g(x) = af(bx+c)+d$ then $(x,y) \mapsto (x/b - c, ay+d)$

6.6 Reflections

Reflection in line	Function	Point
x-axis	$f(x) \mapsto -f(x)$	$(x,y) \mapsto (x,-y)$
y-axis	$f(x) \mapsto f(-x)$	$(x,y) \mapsto (-x,y)$
$y = x$	$f(x) \mapsto f^{-1}(x)$ (inverse function)	$(x,y) \mapsto (y,x)$

7 Proof by induction

For an infinite sequence of mathematical statements, $P(1), P(2), \ldots, P(k), \ldots$:

1. Prove that the first statement is true, $P(1)$.
2. Prove that if a general statement is true then so is the next one, $P(k) \Rightarrow P(k+1)$.

8 Quadratic equations

Discriminant: $\Delta = b^2 - 4ac$

$$\begin{cases} \Delta < 0 & \text{complex conjugate roots} \\ \Delta = 0 & \text{repeated root} \\ \Delta > 0 & \text{two real roots} \end{cases}$$

For complex roots, z_1 and z_2, then $z_1 = z_2^*$

Axis of symmetry passes through vertex.

Form	Roots	Axis of symmetry	Min/max
$ax^2 + bx + c$	$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$-\dfrac{b}{2a}$	
Factorised form: $a(x-p)(x-q)$	p, q	$\dfrac{p+q}{2}$	
Known roots: $x^2 - (\alpha + \beta)x + \alpha\beta$	α, β		
Vertex or turning point form: $a(x-h)^2 + k$	$h \pm \sqrt{-\dfrac{k}{a}}$	h	(h, k)

8.1 Transformation

$$y = A\left(\frac{x}{c} - h\right)^2 + k$$

Vertical scaling	A
Horizontal scaling	c
Vertical translation	k
Horizontal translation	h

9 Exponentials and logs

9.1 Big numbers

million	10^6	duodecillion	10^{39}
billion	10^9	tredecillion	10^{42}
trillion	10^{12}	quattuordecillion	10^{45}
quadrillion	10^{15}	quindecillion	10^{48}
quintillion	10^{18}	sexdecillion	10^{51}
sextillion	10^{21}	septendecillion	10^{54}
septillion	10^{24}	octodecillion	10^{57}
octillion	10^{27}	novemdecillion	10^{60}
nonillion	10^{30}	vigintillion	10^{63}
decillion	10^{33}	centillion	10^{303}
undecillion	10^{36}		

googol	10^{100}
googolplex	$10^{\text{googol}} = 10^{(10^{100})}$

In India, they also use:

lakh	10^5	written as 1,00,000
crore	10^7	written as 1,00,00,000

9.2 Exponentials

$$
\begin{aligned}
a^0 &= 1 \\
a^1 &= a \\
a^2 &= a \times a && (a \text{ squared}) \\
a^3 &= a \times a \times a && (a \text{ cubed})
\end{aligned}
$$

$$
\begin{aligned}
a^{-1} &= \frac{1}{a} & a^{1/2} &= \sqrt{a} \\
a^{-2} &= \frac{1}{a^2} & a^{1/3} &= \sqrt[3]{a} \\
a^{-n} &= \frac{1}{a^n} & a^{1/n} &= \sqrt[n]{a}
\end{aligned}
$$

$\sqrt{\ }$ usually represents the positive square root, and $\pm\sqrt{\ }$ is used for both roots.

$$
\begin{aligned}
a^p \times a^q &= a^{p+q} \\
a^p \div a^q &= a^{p-q} \\
(a^p)^q &= a^{pq} = (a^q)^p \\
a^{p/q} &= \sqrt[q]{a^p} = (\sqrt[q]{a})^p \\
a^p b^p &= (ab)^p
\end{aligned}
$$

By convention, $a^{p^q} = a^{(p^q)}$. For example,:

$$
\begin{aligned}
(2^3)^4 &= 8^4 = 4096 \\
2^{3^4} &= 2^{(3^4)} = 2^{81} \\
&= 2\,417\,851\,639\,229\,258\,349\,412\,352
\end{aligned}
$$

Exponential function: $a \times b^{cx}$

Power law: $a \times x^b$

We write: $\mathrm{e}^x = \exp(x)$.

9.3 Logs

A logarithm is the inverse of an exponential function. If $a^x = b$ then $x = \log_a b$ where $a, b > 0$ and $a \neq 0$.

$$\log_a(a^x) = x = a^{\log_a x}$$

Commonly, $\log x$ is used for $\log_{10} x$ and $\ln x$ for $\log_{\mathrm{e}} x$

$$
\begin{aligned}
\log_{10}(10^x) &= x \\
\log_{10}(10) &= 1 \\
\log_e(\mathrm{e}^x) &= \ln \mathrm{e}^x = x \\
\ln \mathrm{e} &= 1 \\
\log_a(a) &= 1
\end{aligned}
$$

$$\log a^{\log_a x} = x$$
$$10^{\log_{10} x} = x$$
$$\mathrm{e}^{\ln x} = x$$
$$\mathrm{e}^{a \ln b} = b^a$$

$$\log xy = \log x + \log y$$
$$\log \frac{1}{x} = -\log x$$
$$\log \frac{x}{y} = \log x - \log y$$
$$\log x^a = a \log x$$
$$\log \sqrt[a]{x} = \log x^{(1/a)} = \frac{\log x}{a}$$
$$\log_{10} 1 = \log_a 1 = \ln 1 = 0$$
$$\log_{10} 0 = \log_a 0 = \ln 0 = -\infty$$

$$\log_{10} 1 = \log_{10} 10^0 = 0$$
$$\log_{10} 10 = \log_{10} 10^1 = 1$$
$$\log_{10} 100 = \log_{10} 10^2 = 2$$
$$\log_{10} 1000 = \log_{10} 10^3 = 3$$
$$\log_{10} 1\,000\,000 = \log_{10} 10^6 = 6$$

Changing the base:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} = \frac{\ln(x)}{\ln(a)} = \frac{\log_{10}(x)}{\log_{10}(a)}$$

9.4 Log plots

log If $y = ae^{bx}$ then $\ln y = bx + \ln a$.

If this is a straight line then $m = b$ and $c = \ln a$.

log v log If $y = ax^b$ then $\log y = b \log x + \log a$.

If this is a straight line then $m = b$ and $c = \log a$.

9.5 Logs of negative numbers (and complex numbers)

$z = r\,\mathrm{e}^{\mathrm{i}\,\theta}$ so $\ln z = \ln r + \mathrm{i}\,\theta$

$$\ln(-1) = \mathrm{i}\,\pi$$
$$\ln(-2) = \ln(2) + \mathrm{i}\,\pi$$
$$\ln(-x) = \ln(x) + \mathrm{i}\,\pi, \quad x \in \mathbb{R}_{>0}$$

9.6 Exponential decay

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N$$

where N is the amount of something (number of radioactive nuclei) and λ is the decay constant. Solution:

$$N = N_0\,\mathrm{e}^{-\lambda t}$$

The half-life is the time for the amount to decay to half its original value:

$$\text{Half-life:} \quad t_{1/2} = -\frac{\ln(1/2)}{\lambda}$$

9.7 Logistic functions

$$f(x) = \frac{L}{1 + \mathrm{e}^{-k(x-x_0)}} = \frac{L}{1 + C\,\mathrm{e}^{-kx}}$$

L	Maximum asymptotic value; carrying capacity of a population
k	Growth rate
x_0	Midpoint : $f(x_0) = 0.5L$
$C = \mathrm{e}^{kx_0}$	

Asymptote at $y = L$
Point of maximum growth: $(\ln C/k, L/2)$
Initial value: $(0, L/(1 + C))$

Inverse function:

$$x = -\frac{1}{k} \ln\left(\frac{L/y - 1}{C}\right)$$

10 Geometry

10.1 Notation

$A(x, y)$	Point at (x, y) in 2D
O	Origin at $(0, 0)$ in 2D
$[AB]$ or AB	Line segment with endpoints A and B
a, $\lvert a\rvert$, $\lvert AB\rvert$	Length of line AB
a	Side of a polygon
A, $\hat{A}$, $\angle ABC$, $A\hat{B}C$	Angle
A, $\lvert A\rvert$, $\lvert\angle ABC\rvert$, $\lvert A\hat{B}C\rvert$	Measure of an angle
$\triangle ABC$	Triangle
$ABCD$, etc.	Polygon

10.2 Angles

Radians: rad or ^{c} (rarely)

Degrees: $^\circ$ or deg

Conversion:

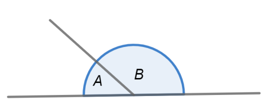
$$\frac{\text{Radian measure}}{\pi} = \frac{\text{Degree measure}}{180^\circ}$$

$$\begin{aligned} 2\pi \,\text{rad} &= 360^\circ \\ \pi \,\text{rad} &= 180^\circ \end{aligned}$$

$$1 \,\text{radian} = \frac{180^\circ}{\pi} = 57.295\,779\,51^\circ$$

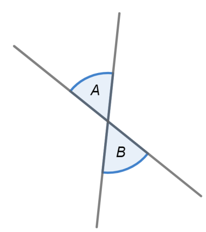
$$1^\circ = \frac{\pi}{180^\circ} = 0.017\,453\,293 \,\text{rad}$$

Angles on a straight line:



$$A + B = 180^\circ$$

Vertically opposite angles:



$$A = B$$

10.3 Points and straight lines

Equation of a straight line (gradient-intercept form):

$$y = mx + c$$

- $m = \tan\theta$ is the slope, where θ is the angle with the x-axis
- c is the y-intercept, where the line cuts the y-axis

General form:

$$ax + by + d = 0$$

where $m = -a/b$ and $c = -d/b$.

Equation from point and gradient:

$$y - y_1 = m(x - x_1)$$

See also the vector equation of a straight line (Section 15).

Two dimensions:

- Two points: (x_1, y_1) and (x_2, y_2)
- Midpoint:
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
- Slope or gradient:
$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$
- $\Delta y = m\Delta x$
- Distance apart: $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Three dimensions:

- Two points: (x_1, y_1, z_1) and (x_2, y_2, z_2)
- Midpoint:
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$
- Distance apart:
$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

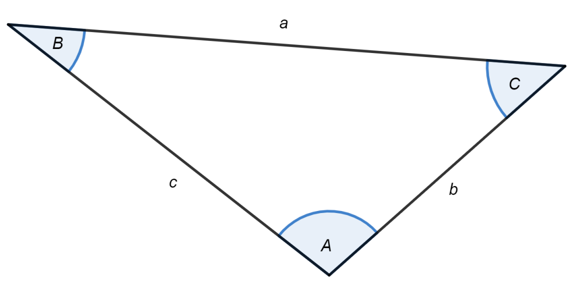
Distance between a straight line and a point:
$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Angle between two lines:
$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

Parallel lines: $m_1 = m_2$

Perpendicular lines: $m_1 m_2 = -1$

10.4 Triangles



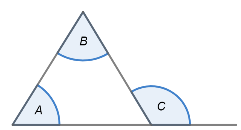
Similar (AAA, i.e., AA): All angles same: $A = B = C$

Congruent (SAS, SSS, ASA or RHS): All angles and all sides same: $A = B = C$ and $a = b = c$. May be mirror images

Identical: All angles and all sides same, and not mirror image

- Angle sum: $180° = \pi\,\text{rad}$
- $|a - b| < c < a + b$
- Perimeter: $a + b + c$

Interior opposite angles



$$A + B = C$$

C is an exterior angle.

Pythagoras theorem

For a right-angled triangle:
$$a^2 = b^2 + c^2$$

where a is the hypotenuse and b and c the other two sides.

Area

$$A = \frac{1}{2}bh = \frac{1}{2}ab\sin C$$

If one vertex is at the origin:
$$A = \frac{1}{2}|x_1 y_2 - x_2 y_1|$$

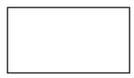
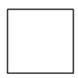
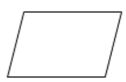
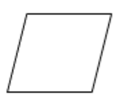
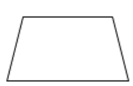
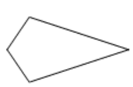
Heron's formula:
$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = (a + b + c)/2$

Area from points:
$$\frac{1}{2}\,\text{abs}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

10.5 Quadrilaterals

Polygon with four sides. Not necessarily regular. Angle sum is $360°$.

Quadrilateral		Area	Perimeter
	**Rectangle** Opposite sides equal and parallel All angles the same size ($90°$) Diagonals bisect each other Diagonal creates two congruent triangles	ab	$2(a+b)$
	**Square** All sides equal Opposite sides parallel All angles the same size ($90°$) Diagonals bisect each other at $90°$ Diagonal creates two congruent triangles	a^2	$4a$
	**Parallelogram** Opposite sides equal and parallel Opposite angles equal Adjacent angles sum to $180°$ Diagonals bisect each other Diagonal creates two congruent triangles	$ab\sin C = bh$	$2(a+b)$
	**Rhombus** All sides equal Opposite sides parallel Opposite angles equal Adjacent angles sum to $180°$ Diagonals bisect each other at $90°$ Diagonal creates two congruent triangles	$a^2\sin C = ah$	$4a$
	**Trapezium** One pair of opposite sides parallel	$\left(\frac{a+b}{2}\right)h$	$a+b+c+d$
	**Kite** Two pairs of adjacent sides equal		

10.6 Regular polygons

All sides and all interior angles are equal.

Polygon	No of sides
Equilateral triangle	3
Square	4
Pentagon	5
Hexagon	6
Septagon	7
Octagon	8
Nonagon	9
Decagon	10

Interior angle: $\dfrac{180(n-2)}{n}^{\circ}$

Area of a hexagon: $\dfrac{3}{2}\sqrt{3}l^2$ (l = side length)

10.7 Circles

With centre (h, k) and radius r:

$$(x-h)^2 + (y-k)^2 = r^2$$

With centre $(-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Circumference: $2\pi r$

Area: πr^2

Tangent to a circle through a point:

$$(x-h)(x_1-h) + (y-k)(y_1-k) = r^2$$
$$xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$$

An angle at the centre of a circle standing on an arc is twice the angle at any point on the circle standing on the same arc. All angles standing on the same arc are equal.

Angle in a semicircle is 90°. Converse: If the angle on a chord is 90°, the chord is a diameter.

Angle between a tangent and the radius is 90°.

If two circles intersect at only one point, the two centres and the intersection are collinear.

The perpendicular bisector of a chord bisects the chord.

Sectors

Length of an arc:

$$l = \begin{cases} r\theta, & \theta \text{ in rad} \\ 2\pi r \dfrac{\theta}{360^\circ} = \pi r \dfrac{\theta}{180^\circ}, & \theta \text{ in degrees} \end{cases}$$

Area of a sector:

$$A = \begin{cases} \dfrac{1}{2} r^2 \theta, & \theta \text{ in rad} \\ \pi r^2 \dfrac{\theta}{360^\circ}, & \theta \text{ in degrees} \end{cases}$$

Perimeter of an arc: $r\theta + 2r$ (θ in rad)

Cyclic quadrilaterals

- Vertices lie on a circle, called the circumcircle.
- Opposite angles sum to 180°.
- Since the opposite angles in a parallelogram are equal, the only *parallelograms* that can be inscribed in a circle are rectangles (other quadrilaterals are possible).

10.8 Ellipses

Equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

centred at the origin where a and b are the axes.

Area: $A = \pi ab$

Eccentricity:

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

There is no simple equation for the perimeter of an ellipse.

10.9 Solids

Regular solids

Euler's formula:

$$V - E + F = 2$$

where V is the number of vertices, E the number of edges and F the number of faces.

Volume of a cube: l^3

Solid	Vertices	Edges	Faces
Tetrahedron 4 equilateral triangles	4	6	4
Cube 6 squares	8	12	6
Octahedron 8 equilateral triangles	6	12	8
Dodecahedron 12 regular pentagons	20	30	12
Icosahedron 20 equilateral triangles	12	30	20

Other solids

Solid	Volume	Curved surface area	Total surface area
Cuboid	lwh	–	$2(lw + wh + hl)$
Sphere	$\frac{4}{3}\pi r^3$	$2\pi r^2$	$4\pi r^2$
Hemisphere	$\frac{2}{3}\pi r^3$	$2\pi r^2$	$3\pi r^2$
Cone	$\frac{1}{3}\pi r^2 h$	$\pi r l$ Slant length $l = \sqrt{r^2 + h^2}$	$\pi r l + \pi r^2$
Truncated cone (frustum of a cone)	$\frac{1}{3}\pi(r_1^2 + r_1 r_2 + r_2^2)h$		
Cylinder	$\pi r^2 h$	$2\pi r h$	$2\pi r h + 2\pi r^2$
Prism	Ah	–	
Tetrahedron a is the side	$\dfrac{a^3}{6\sqrt{2}}$	–	$\sqrt{3}a^3$
Pyramid (base of any shape)	$\frac{1}{3}Ah$	–	
Frustum of a pyramid	$\frac{1}{3}h(A_1 + A_2 + \sqrt{A_1 A_2})$	–	

11 Trigonometry

Mnemonic for remembering trig ratios:

SOH CAH TOA

$$\sin\theta = \frac{O}{H} \qquad \cos\theta = \frac{A}{H} \qquad \tan\theta = \frac{O}{A}$$

$$\begin{aligned} \cos(-\theta) &\equiv \cos\theta \\ \sin(-\theta) &\equiv -\sin\theta \\ \tan(-\theta) &\equiv -\tan\theta \end{aligned}$$

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \qquad \cot\theta \equiv \frac{1}{\tan\theta}$$

$$\sec\theta \equiv \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta = \csc\theta \equiv \frac{1}{\sin\theta}$$

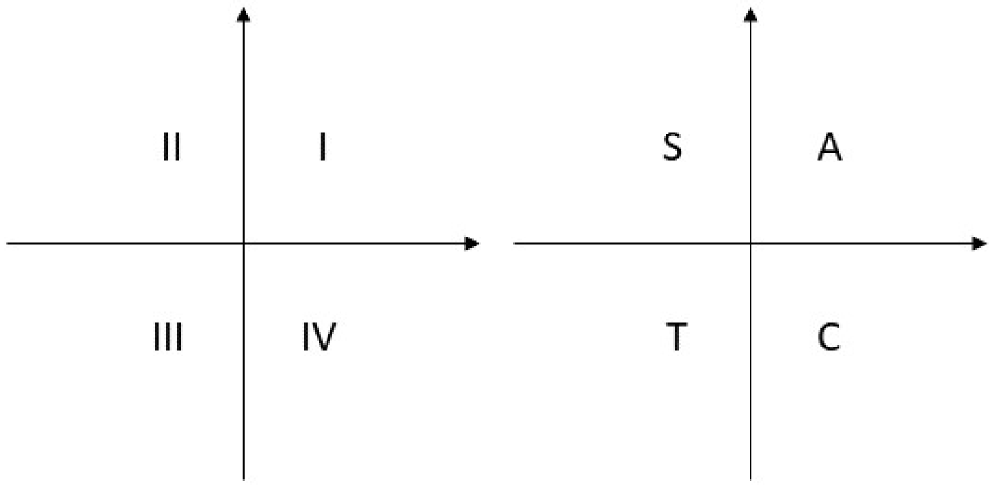
Inverse functions:

$$\begin{aligned} \theta &= \arcsin\frac{O}{H} = \sin^{-1}\frac{O}{H} \\ &= \arccos\frac{A}{H} = \cos^{-1}\frac{A}{H} \\ &= \arctan\frac{O}{A} = \tan^{-1}\frac{O}{A} \end{aligned}$$

11.1 Special angles

Degrees	Radians	Cosine	Sine	Tangent
0°	0	1	0	0
30°	$\pi/6$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}/3 = 1/\sqrt{3}$
45°	$\pi/4$	$\sqrt{2}/2 = 1/\sqrt{2}$		1
60°	$\pi/3$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}$
90°	$\pi/2$	0	1	undefined, $\pm\infty$
120°	$2\pi/3$	$-1/2$	$\sqrt{3}/2$	$-\sqrt{3}$
135°	$3\pi/4$	$-\sqrt{2}/2 = -1/\sqrt{2}$	$\sqrt{2}/2 = 1/\sqrt{2}$	-1
150°	$5\pi/6$	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}/3 = -1/\sqrt{3}$
180°	π	-1	0	0
210°	$7\pi/6$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}/3 = 1/\sqrt{3}$
225°	$5\pi/4$	$-\sqrt{2}/2 = -1/\sqrt{2}$		1
240°	$4\pi/3$	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}$
270°	$3\pi/2$	0	-1	undefined, $\pm\infty$
300°	$5\pi/3$	$1/2$	$-\sqrt{3}/2$	$-\sqrt{3}$
315°	$7\pi/4$	$\sqrt{2}/2 = 1/\sqrt{2}$	$-\sqrt{2}/2 = -1/\sqrt{2}$	-1
330°	$11\pi/6$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}/3 = 1/\sqrt{3}$
$360^\circ = 0^\circ$	2π	1	0	0

11.2 Angles in other quadrants



Mnemonic: CAST

Quadrant	Positive ratios	Range (rad)	Range (degrees)
I	All positive	$]0, \pi/2[$	$]0°, 90°[$
II	Sine positive $\sin(180° - \theta) = \sin(\pi - \theta) = +\sin\theta$ $\cos(180° - \theta) = \cos(\pi - \theta) = -\cos\theta$ $\tan(180° - \theta) = \tan(\pi - \theta) = -\tan\theta$	$]\pi/2, \pi[$	$]90°, 180°[$
III	Tangent positive $\sin(180° + \theta) = \sin(\pi + \theta) = -\sin\theta$ $\cos(180° + \theta) = \cos(\pi + \theta) = -\cos\theta$ $\tan(180° + \theta) = \tan(\pi + \theta) = +\tan\theta$	$]\pi, 3\pi/2[$	$]180°, 270°[$
IV	Cosine positive $\sin(360° - \theta) = \sin(2\pi - \theta) = -\sin\theta$ $\cos(360° - \theta) = \cos(2\pi - \theta) = +\cos\theta$ $\tan(360° - \theta) = \tan(2\pi - \theta) = -\tan\theta$	$]3\pi/2, 2\pi[$	$]270°, 360°[$
$0 \leq \theta \leq \pi$ (90°)			

For any θ:

$$\sin(\theta + 2n\pi) = \sin(\theta), \qquad n \in \mathbb{Z}$$
$$\cos(\theta + 2n\pi) = \cos(\theta), \qquad n \in \mathbb{Z}$$
$$\tan(\theta + n\pi) = \tan(\theta), \qquad n \in \mathbb{Z}$$

Phase shift (changes the trig ratio):

$$\sin(\pi/2 + \theta) = +\cos\theta$$
$$\cos(\pi/2 + \theta) = -\sin\theta$$
$$\tan(\pi/2 + \theta) = -\cot\theta$$

11.3 Identities

Pythagorean identities:

$$\begin{aligned}\cos^2 A + \sin^2 A &\equiv 1\\ 1 + \tan^2 A &\equiv \sec^2 A\\ 1 + \cot^2 A &\equiv \csc^2 A\end{aligned}$$

$$\begin{aligned}\sin(A+B) &\equiv \sin A\cos B + \cos A\sin B\\ \sin(A-B) &\equiv \sin A\cos B - \cos A\sin B\\ \sin(2A) &\equiv 2\sin A\cos A\\ &\equiv \frac{2\tan A}{1+\tan^2 A}\end{aligned}$$

$$\begin{aligned}\cos(A+B) &\equiv \cos A\cos B - \sin A\sin B\\ \cos(A-B) &\equiv \cos A\cos B + \sin A\sin B\\ \cos(2A) &\equiv \cos^2 A - \sin^2 A\\ &\equiv 2\cos^2 A - 1\\ &\equiv 1 - 2\sin^2 A\\ &\equiv \frac{1-\tan^2 A}{1+\tan^2 A}\end{aligned}$$

$$\begin{aligned}\tan(A+B) &\equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}\\ \tan(A-B) &\equiv \frac{\tan A - \tan B}{1 + \tan A\tan B}\\ \tan(2A) &\equiv \frac{2\tan A}{1-\tan^2 A}\end{aligned}$$

$$\begin{aligned}\sin A + \sin B &\equiv 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\\ \sin A - \sin B &\equiv 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\\ \cos A + \cos B &\equiv 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}\\ \cos A - \cos B &\equiv -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}\end{aligned}$$

$$\begin{aligned}\cos A + \sin A &\equiv \sqrt{2}\sin\left(\frac{\pi}{4} + A\right)\\ \cos A - \sin A &\equiv \sqrt{2}\sin\left(\frac{\pi}{4} - A\right)\end{aligned}$$

$$\begin{aligned}2\cos A\cos B &\equiv \cos(A+B) + \cos(A-B)\\ 2\sin A\cos B &\equiv \sin(A+B) + \sin(A-B)\\ 2\sin A\sin B &\equiv \cos(A+B) - \cos(A-B)\\ \sin^2 A &\equiv \frac{1}{2}(1-\cos 2A)\\ \cos^2 A &\equiv \frac{1}{2}(1+\cos 2A)\end{aligned}$$

de Moivre:

$$(\cos\theta + \mathrm{i}\sin\theta)^n \equiv \cos(n\theta) + \mathrm{i}\sin(n\theta)$$

11.4 Triangle rules

Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

11.5 Periodic functions

$$\begin{aligned}y = f(x) &= a\sin(b(x-c)) + d\\ &= a\cos\left(b(x-c) - \frac{\pi}{2}\right) + d\end{aligned}$$

Midline: $y = d$
(equation of principal axis)

Amplitude: $|a|$
(height above midline)

Period: $T = \dfrac{2\pi}{b}$
(distance between maxima or minima, or for one revolution)

Angular frequency: $\omega = \dfrac{b}{2\pi} = \dfrac{1}{T}$

Phase shift: c
($c > 0$ to the right and $c < 0$ to the left – note minus sign in equation above)

Vertical translation: d
($d > 0$ up and $d < 0$ down)

Periodic function of time

$$\begin{aligned} y(t) &= a\sin(\omega t + \phi) \\ &= a\sin(2\pi f t + \phi) \\ y(x,t) &= a\sin(kx + \omega t + \phi) \end{aligned}$$

where k is the wavenumber (rad/metre), ω is angular frequency (rad/second), t is time (seconds) and ϕ is the phase difference (rad). Since sin is dimensionless, a and y have the same units.

$$k = \frac{\omega}{v} = \frac{2\pi f}{v} = \frac{2\pi}{\lambda}$$

where λ is the wavelength (metres) and v is the linear speed (metres/second).

- Out of phase: $\phi \neq 0$
- In phase: $\phi = 0$
- Antiphase: $\phi = \pi\,\text{rad} = 180°$

12 Calculus

12.1 Limits

Limit of $f(x)$ as x approaches a from values less than a:

$$\lim_{x\to a^-} f(x)$$

Limit of $f(x)$ as x approaches a from values more than a:

$$\lim_{x\to a^+} f(x)$$

$\lim_{x\to a} f(x)$ is defined only if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

$$\begin{aligned} \lim_{x\to a}(f(x)+g(x)) &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\ \lim_{x\to a}(f(x)-g(x)) &= \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \\ \lim_{x\to a}(f(x)\times g(x)) &= \lim_{x\to a} f(x) \times \lim_{x\to a} g(x) \\ \lim_{x\to a}(kf(x)) &= k\lim_{x\to a} f(x) \\ \lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ if } \lim_{x\to a} g(x) \neq 0 \end{aligned}$$

L'Hôpital's rule

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\pm\infty$, then

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)}$$

12.2 Derivatives

If $y = f(x)$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Second derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f''(x), \text{ etc.}$$

nth derivative:

$$\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = f^{(n)}(x)$$

If $x = f(t)$ then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}, \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \ddot{x}, \text{ etc.}$$

12.3 Fundamental theorem of calculus

Indefinite integral:

$$\int f(x)\,\mathrm{d}x = F(x)$$

Definite integral:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

where $F'(x) = f(x)$.

12.4 Standard derivatives and integrals

Function, $f(x)$	Derivative, $f'(x)$	Integral, $\int f(x)\,\mathrm{d}x = F(x)$
$f(ax+b)$	$af'(ax+b)$	$\frac{1}{x}F(ax+b)$
c	0	cx
ax	a	$\frac{1}{2}ax^2$
x^n	nx^{n-1}	$\frac{1}{n+1}x^{n+1}$, $n \neq -1$
ax^n	anx^{n-1}	$\frac{a}{n+1}x^{n+1}$, $n \neq -1$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\ln\lvert x\rvert$
$\frac{1}{ax+b}$	$-\frac{a}{(ax+b)^2}$	$\frac{1}{a}\ln\lvert ax+b\rvert$
$\frac{1}{x^2+a^2}$	$-\frac{2x}{(x^2+a^2)^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
e^x	e^x	e^x
e^{-x}	$-\mathrm{e}^{-x}$	$-\mathrm{e}^{-x}$
e^{ax}	$a\,\mathrm{e}^{ax}$	$\frac{1}{a}\,\mathrm{e}^{ax}$
$a\,\mathrm{e}^{bx+c}$	$ab\,\mathrm{e}^{bx+c}$	$\frac{1}{b}a\,\mathrm{e}^{bx+c}$
a^x	$a^x\ln a$	$\frac{a^x}{\ln a}$, $a > 0$
a^{bx}	$ba^{bx}\ln a$	$\frac{a^{bx}}{b\ln a}$, $a > 0$

Function, $f(x)$	Derivative, $f'(x)$	Integral, $\int f(x)\,\mathrm{d}x = F(x)$
a^{bx+c}	$ba^{bx+c}\ln a$	$\dfrac{a^{bx+c}}{b\ln a},\ a>0$
$\ln x$	$\dfrac{1}{x}$	$x(\ln(x)-1)$
$\ln ax$	$\dfrac{1}{x}$	$x(\ln(ax)-1)$
$\log_a x$	$\dfrac{1}{x\ln a}$	$\dfrac{x(\ln(x)-1)}{\ln a}$
$\sin x$	$\cos x$	$-\cos x$
$\sin ax$	$a\cos ax$	$-\dfrac{1}{a}\cos ax$
$\sin(ax+b)$	$a\cos(ax+b)$	$-\dfrac{1}{a}\cos(ax+b)$
$\cos x$	$-\sin x$	$\sin x$
$\cos ax$	$-a\sin ax$	$\dfrac{1}{a}\sin ax$
$\cos(ax+b)$	$-a\sin(ax+b)$	$\dfrac{1}{a}\sin(ax+b)$
$\tan x$	$\dfrac{1}{\cos^2 x}=\sec^2 x$	$-\ln(\lvert\cos x\rvert)$
$\tan ax$	$\dfrac{b}{\cos^2 bx}=b\sec^2 bx$	$-\dfrac{1}{a}\ln(\lvert\cos ax\rvert)$
$\sec x=\dfrac{1}{\cos x}$	$\sec x\tan x$	$\ln(\tan x+\sec x)$
$\csc x=\dfrac{1}{\sin x}$	$-\csc x\cot x$	$-\ln(\cot x+\csc x)$
$\cot x=\dfrac{1}{\tan x}$	$-\csc^2 x$	$\ln(\sin x)$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sqrt{1-x^2}+x\sin^{-1}x$
$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$	$-\sqrt{1-x^2}+x\cos^{-1}x$
$\arctan x$	$\dfrac{1}{1+x^2}$	$x\tan^{-1}x-\dfrac{1}{2}\ln(x^2+1)$
$\arcsin\dfrac{x}{a}$	$\dfrac{1}{\sqrt{a^2-x^2}}$	

Function, $f(x)$	Derivative, $f'(x)$	Integral, $\int f(x)\,\mathrm{d}x = F(x)$
$\arccos \frac{x}{a}$	$-\frac{1}{\sqrt{a^2 - x^2}}$	
$\arctan \frac{x}{a}$	$\frac{a}{a^2 + x^2}$	

Note: All integrals above are indefinite and need a constant, C.

12.5 Differentiation rules

Reciprocal:

$$\frac{1}{\mathrm{d}y/\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}y}$$

Rule	Function, $f(x)$	Derivative, $f'(x)$
Addition	$u(x) + v(x)$	$u'(x) + v'(x)$
Product rule	uv	$uv' + vu'$
Quotient rule ($v(x) \neq 0$)	$\frac{u}{v}$	$\frac{1}{v^2}(vu' - uv')$
Chain rule	$y = g(u)$ and $u = f(x)$	$\frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$
	$F(x) = f \circ g(x)$	$F'(x) = f'(g(x)) \times g'(x)$

12.6 Stationary points

- Maxima, minima and horizontal points of inflection: $f'(x) = 0$
- Point of inflection: the slope changes from concave (concave down) to convex (concave up) or vice versa. The curve is not necessarily horizontal here.

Type	$f'(x)$	$f''(x)$
Maximum (concave down)	0 $(+0-)$	negative (or 0)
Minimum (concave up)	0 $(-0+)$	positive (or 0)
Stationary point of inflection (saddle)	0	0 – changes sign either side of zero
Non-stationary point of inflection	non-zero	0 – changes sign either side of zero

12.7 Kinematics

$$v(t) = \frac{\mathrm{d}s}{\mathrm{d}t}$$
$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s}$$

Final displacement: $\int_a^b v(t)\,\mathrm{d}t$

Total distance: $\int_a^b |v(t)|\,\mathrm{d}t$

12.8 Integration

Area under a curve:

$$A = \int_a^b y\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

Volume of a solid of revolution about x-axis:

$$V = \int_a^b \pi y^2\,\mathrm{d}x = \int_a^b \pi(f(x))^2\,\mathrm{d}x$$

Average of a function between $x = a$ and $x = b$:

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

Trapezoidal rule for area between $x = a$ and $x = b$:

$$\frac{1}{2}h[y_0 + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1}) + y_n]$$

where h is the height of the trapezium, i.e., the horizontal width between y values:

$$h = \frac{b-a}{n}$$

Reverse chain rule

$$\int u' \times u'(v)\,\mathrm{d}x = u(v)$$
$$\int u(v) \times v'\,\mathrm{d}x = \int u(v)\,\mathrm{d}v$$

Integration by parts

$$\int uv\,\mathrm{d}x = u\int v\,\mathrm{d}x - \int u'\left(\int v\,\mathrm{d}x\right)\mathrm{d}x$$
$$\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$$
$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

12.9 Partial derivatives

Chain rule:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

for $u = u(x, y)$, $x = x(t)$ and $y = y(t)$

Implicit differentiation:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial u}{\partial x}$$

for $u = u(x, y)$

13 Statistics

13.1 Types of data

Categorical v numerical data

- Categorical data *cannot* be represented by a number: shapes (triangle, circle, cross), gender
- Numerical data can be represented by numbers: number of biscuits in a packet, heights

Ordinal categorical data v nominal categorical data

- Ordinal data can be ordered in some way: sizes (small, medium, large)

- Nominal data *cannot* be ordered: type of vehicle (bus, lorry, truck)

Numerical data can be discrete or continuous

- Discrete data take only certain values in discrete steps: number of houses in a street, electric charge $(-2, -1, 0, +1, +2)$
- Continuous data can take on any values in a range and are usually from a measurement: weights, lengths, voltage, current

13.2 Central tendency

Mean of a population N:

$$\mu = \frac{\sum_1^N x_i}{N} = \frac{x_1 + x_2 + x_3 + \cdots + x_N}{N}$$

Mean of a sample n:

$$\bar{x} = \frac{\sum_1^n x_i}{n} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

Mode is the most common entry.

Median is the middle entry for an odd number of entries, or the average of the two entries around the middle for even number of entries. Median is Q_2.

The mean is best for a symmetrical distribution because it includes all of the data. The median is best for a skewed distribution because it reduces the effect of outliers. The mode is used for nominal data.

13.3 Dispersion

Variance of a population:

$$\mathrm{Var} = \sigma^2 = \frac{\sum (x-\mu)^2}{N} = \frac{\sum x^2}{N} - \mu^2$$

Standard deviation of a population:

$$\sigma = \sqrt{\frac{\sum (x-\mu)^2}{N}} = \sqrt{\frac{\sum x^2}{N} - \mu^2}$$

Unbiased estimate of population variance:

$$s_{n-1}^2 = \frac{\sum (x-\bar{x})^2}{n-1} = \frac{\sum x^2}{n-1} - \frac{n}{n-1}\bar{x}^2$$

Standard deviation of a sample:

$$\sigma = s_{n-1} = \sqrt{\frac{\sum (x-\bar{x})^2}{n-1}}$$

Interquartile range: $\mathrm{IQR} = Q_3 - Q_1$

Outlier: more than $1.5 \times \mathrm{IQR}$ below Q_1 or above Q_3.

13.4 Grouped data

Population size:

$$n = \sum_{i=1}^{i=k} f_i$$

where f_i is the frequency of event x_i. (All following sums over i have the same limits.)

Mean:

$$\mathrm{E}(x) = \mu = \frac{\sum f_i x_i}{\sum f_i} = \frac{\sum f_i x_i}{n}$$

Standard deviation:

$$\begin{aligned}\sigma &= \sqrt{\frac{\sum f_i (x_i - \mu)^2}{n}} = \sqrt{\frac{\sum f_i (x_i - \mu)^2}{\sum f_i}} \\ &= \sqrt{\frac{\sum f_i x_i^2}{n} - \mu^2} = \sqrt{\frac{\sum f_i x_i^2}{\sum f_i} - \mu^2}\end{aligned}$$

Variance: $\mathrm{Var} = \sigma^2$

Unbiased estimate of population variance:

$$s_{n-1}^2 = \frac{\sum f_i (x-\bar{x})^2}{n-1} = \frac{\sum f_i x^2}{n-1} - \frac{n}{n-1}\bar{x}^2$$

Modal class is class with highest frequency.

$$\mathrm{Median} = L + \frac{n/2 - B}{G} \times W$$

where L is the lower boundary for the median group, n is the total number of entries, B is the cumulative frequency of the groups before the median group, G is the frequency of the median group and W is the width of each group.

$$Q_1 = L + \frac{n/4 - B}{G} \times W$$

$$Q_3 = L + \frac{3n/4 - B}{G} \times W$$

13.5 Errors

Measurement error: $M_{\text{actual}} - M_{\text{estimate}}$

Absolute measurement error: $|M_{\text{actual}} - M_{\text{estimate}}|$

Percentage error:

$$\left| \frac{M_{\text{actual}} - M_{\text{estimate}}}{M_{\text{estimate}}} \right| \times 100\%$$

Margin of error is $1/\sqrt{n}$

Standard error of the mean: $\sigma/\sqrt{n}$

Confidence interval = percentage ± margin of error as percentage

z-score (known variance)

Population:

$$z = \frac{x - \mu}{\sigma}$$

where x is a measurement.

Sample:

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

Confidence interval: $\bar{x} \pm \frac{z\sigma}{\sqrt{n}}$

Use inverse normal to get z from a probability.

Student's t (unknown variance)

$$t = \frac{\bar{x} - \mu}{s_{n-1}/\sqrt{n}}$$

Confidence interval: $\bar{x} \pm \frac{ts_{n-1}}{\sqrt{n}}$

13.6 Skewness

- Positive: More entries on the left, tail on right
- Zero: Symmetric
- Negative: More entries on the right, tail on left

13.7 Correlation

Correlation does not imply causation.

Pearson's correlation coefficient or sample product moment correlation coefficient:

$$\begin{aligned} r &= \frac{n \sum x_i y_i - \sum x_i \sum y_i}{\sqrt{(n \sum x_i^2 - (\sum x_i)^2)(n \sum y_i^2 - (\sum y_i)^2)}} \\ &= \frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sqrt{\left(\sum x_i^2 - n\bar{x}^2\right)\left(\sum y_i^2 - n\bar{y}^2\right)}} \end{aligned}$$

-1	Perfect negative correlation
-0.70	Strong negative correlation
-0.50	Moderate negative correlation
-0.30	Weak negative correlation
0	No correlation
$+0.30$	Weak positive correlation
$+0.50$	Moderate positive correlation
$+0.70$	Strong positive correlation
$+1$	Perfect positive correlation

Coefficient of determination: r^2

Equations of regression lines:

$$x - \bar{x} = \left(\frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum y_i^2 - n\bar{y}^2} \right) (y - \bar{y})$$

$$y - \bar{y} = \left(\frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum x_i^2 - n\bar{x}^2} \right) (x - \bar{x})$$

14 Probability

- Experiment: A process used to obtain an observation.
- Trial: A run of an experiment.
- Outcome: A possible result of an experiment.
- Event: An outcome or set of outcomes.
- Sample space: The set of all possible outcomes of an experiment, U.

Probability of an event A:

$$\mathrm{P}(A) = \frac{n(A)}{n(U)}$$

Complementary events:

$$\mathrm{P}(A) + \mathrm{P}(A') = 1$$

Addition rule:

$$\begin{aligned}\mathrm{P}(A \cup B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)\\ \mathrm{P}(A \cap B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B)\end{aligned}$$

Conditional probability (probability that A will occur given that B has occurred):

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{n(A \cap B)}{n(B)}$$

Multiplication rule:

$$\mathrm{P}(A \cap B) = \mathrm{P}(A|B)\,\mathrm{P}(B) = \mathrm{P}(B|A)\,\mathrm{P}(A)$$

Total probability:

$$\mathrm{P}(A) = \mathrm{P}(A|B)\,\mathrm{P}(B) + \mathrm{P}(A|B')\,\mathrm{P}(B')$$

Bayes theorem:

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(B|A)\,\mathrm{P}(A)}{\mathrm{P}(B)}$$

14.1 Mutually exclusive events

Two events A and B are mutually exclusive if they cannot occur at the same time:

$$\begin{aligned}n(A \cap B) &= 0\\ \mathrm{P}(A \cap B) &= 0\\ \mathrm{P}(A \cup B) &= \mathrm{P}(A) + \mathrm{P}(B)\\ \mathrm{P}(A|B) &= 0\end{aligned}$$

14.2 Dependent events

Two events A and B are dependent if the occurrence of one depends on the occurrence of the other (like taking coloured balls from a bag *without* replacement).

$$\begin{aligned}\mathrm{P}(B|A) &\neq \mathrm{P}(B)\\ \mathrm{P}(A|B) &\neq \mathrm{P}(A)\\ \mathrm{P}(B \cap A) &\neq \mathrm{P}(B)\,\mathrm{P}(A)\end{aligned}$$

14.3 Independent events

Two events A and B are independent if the occurrence of one is not affected by the occurrence of the other (like taking coloured balls from a bag *with* replacement).

$$\begin{aligned}\mathrm{P}(B|A) &= \mathrm{P}(B) = \mathrm{P}(B|A')\\ \mathrm{P}(A|B) &= \mathrm{P}(A) = \mathrm{P}(A|B')\\ \mathrm{P}(B \cap A) &= \mathrm{P}(B)\,\mathrm{P}(A)\end{aligned}$$

14.4 Combinations and permutations

Permutations are ordered:

$${}^{n}P_{x} = \frac{n!}{(n-x)!}$$

Combinations are not:

$$\binom{n}{x} = {}^{n}C_{x} = \frac{n!}{x!(n-x)!}$$

where n is the total population and x is the number chosen at random.

Binomial theorem

$$\begin{aligned}(a+b)^n &= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \\ &= a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{r} a^{n-r} b^r \\ &\quad + \cdots + b^n\end{aligned}$$

14.5 Variance and expected values

$$\begin{aligned}\mathrm{Var}(X) &= \sigma^2 \\ &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \\ &= \mathrm{E}(X^2) - \mu^2 \\ &= \mathrm{E}[(X - \mathrm{E}(X))^2] \\ &= \mathrm{E}[(X-\mu)^2]\end{aligned}$$

14.6 Discrete random variables

Sum of probabilities:

$$\sum_i \mathrm{P}(X = x_i) = \sum_i p_i = 1$$

Expected value:

$$\mathrm{E}(X) = \mu = \sum_i x_i \ \mathrm{P}(X = x_i) = \sum_i x_i p_i$$

A game is fair if $\mathrm{E}(X) = 0$.

Variance

$$\begin{aligned}\mathrm{Var}(X) &= \sigma^2 \\ &= \sum_i (x_i - \mu)^2 \ \mathrm{P}(X = x_i) \\ &= \sum_i (\mathrm{P}(X = x_i) x_i^2) - \mu^2 \\ &= \sum_i (x_i - \mu)^2 \, p_i \\ &= \sum_i (p_i x_i^2) - \mu^2\end{aligned}$$

Cumulative distribution function (CDF)

$$F(x) = \mathrm{P}(X \le x) = \sum_{x_i \le x} \mathrm{P}(X = x_i)$$

$$\begin{aligned}\mathrm{P}(a < X \le b) &= F(b) - F(a) \\ \mathrm{P}(a < X) &= 1 - F(a)\end{aligned}$$

Combinations of random variables

If mean/mode/median for data set $\{x\}$ is μ and standard deviation is σ, then for $\{ax + b\}$ mean/mode/median is $a\mu + b$ and standard deviation is $|a|\sigma$.

$$\begin{aligned}\mathrm{E}(aX + b) &= a\,\mathrm{E}(X) + b \\ \mathrm{Var}(aX + b) &= a^2\,\mathrm{Var}(X)\end{aligned}$$

For independent random variables:

$$\begin{aligned}\mathrm{E}(aX + bY) &= a\,\mathrm{E}(X) + b\,\mathrm{E}(Y) \\ \mathrm{E}(aX - bY) &= a\,\mathrm{E}(X) - b\,\mathrm{E}(Y) \\ \mathrm{E}(XY) &= \mathrm{E}(X)\,\mathrm{E}(Y) \\ \mathrm{Var}(X + Y) &= \mathrm{Var}(X) + \mathrm{Var}(Y) \\ \mathrm{Var}(aX + bY) &= a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) \\ \mathrm{Var}(aX - bY) &= a^2\,\mathrm{Var}(X) - b^2\,\mathrm{Var}(Y)\end{aligned}$$

14.7 Central limit theorem

The means of samples of size n of a population with mean μ and standard deviation σ can be modelled by a normal distribution $N(\mu, \sigma^2/n)$, provided that $n \ge 30$. This holds regardless of whether the source population is normal or skewed.

14.8 General distribution functions

$\mathrm{E}(X)=\mu$	$\mathrm{Var}(X)=\sigma^2$	Mode	Median
Normal (see below): $X \sim \mathrm{N}(\mu,\sigma^2)$, $\mathrm{P}(a<X<b)=\dfrac{1}{\sigma\sqrt{2\pi}}\int_a^b \mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,\mathrm{d}x$			
μ	σ^2	μ	μ
Binomial: $X \sim \mathrm{B}(n,p)$, $\mathrm{P}(X=x)=\binom{n}{x}p^x q^{n-x}=\binom{n}{x}p^x(1-p)^{n-x}$, $x=0,1,2,3,\ldots,n$			
np	$npq=np(1-p)$	x with highest probability	
Bernoulli: $X \sim \mathrm{B}(1,p)$, $\mathrm{P}(X=x)=p^x q^{1-x}$ for $x=0,1$ (i.e. the binomial with $n=1$)			
p	$pq=p-p^2$		
Geometric: $X \sim \mathrm{Geo}(p)$, $\mathrm{P}(X=x)=p(1-p)^{x-1}$ for $x=1,2,3,\ldots$			
$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}=\dfrac{q}{p^2}$		
Negative binomial: $X \sim \mathrm{NB}(r,p)$, $\mathrm{P}(X=x)=\binom{x-1}{r-1}p^r(1-p)^{x-r}$ for $x=r,r+1,r+2,\ldots$			
$\dfrac{r}{p}$	$\dfrac{r(1-p)}{p^2}=\dfrac{rq}{p^2}$		
Poisson: $X \sim \mathrm{Po}(\mu)$, $\mathrm{P}(X=x)=\dfrac{\mu^x\,\mathrm{e}^{-\mu}}{x!}$, $x=0,1,2,3,\ldots$			
μ	μ		

14.9 Probability density functions

For independent continuous random variables:

$$f(x,y)=f_X(x)f_Y(y)$$

If the random variable X has probability density function f, then the random variable $Y=aX+b$ has probability density function defined by

$$g(x)=\frac{1}{|a|}f\left(\frac{x-b}{a}\right)$$

The probability density function of the random variable kX is defined by

$$f_{kX}(x)=\frac{1}{k}f_X\left(\frac{x}{k}\right).$$

The probability density function of the random variable $X+Y$ is defined by

$$f_{X+Y}(x)=\int_{-\infty}^{\infty} f_X(t)f_Y(x-t)\,\mathrm{d}t.$$

14.10 Continuous random variables

Uses a probability density function, $f(x)$.

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x &= 1 \\
\mathrm{P}(a \le X \le b) &= \mathrm{P}(a \le X < b) \\
&= \mathrm{P}(a < X \le b) = \mathrm{P}(a < X < b) \\
&= \int_a^b f(x)\,\mathrm{d}x \\
\mathrm{P}(X = x) &= 0 \quad \text{(only defined for a range)} \\
\mathrm{E}(X) = \mu &= \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x \\
\mathrm{Var}(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x - \mu^2
\end{aligned}
$$

Cumulative:

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t \text{ and } F'(x) = f(x)$$

Mode is a local max of the PDF, $f(x)$.

Median defined by:

$$\int_{-\infty}^{m} f(x)\,\mathrm{d}x = F(m) = \frac{1}{2}$$

Similarly for quartiles.

14.11 Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{2}$$

1σ	$\mathrm{P}(\mu - \sigma < X < \mu + \sigma) = 0.683$	68%
2σ	$\mathrm{P}(\mu - 2\sigma < X < \mu + 2\sigma) = 0.954$	95%
3σ	$\mathrm{P}(\mu - 3\sigma < X < \mu + 3\sigma) = 0.997$	99.7%

Standard normal: $\mu = 0$, $\sigma = 1$

$$f(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}x^2} \tag{3}$$

14.12 Hypothesis testing

Type I error: Reject null hypothesis when true. False positive.

Type II error: Accept null hypothesis when not true. False negative.

Test statistic for H_0 ($\rho = 0$):

$$t = r\sqrt{\frac{n-2}{1-r^2}}$$

15 Vectors

Vectors are written as $\vec{a}$ or $\mathbf{a}$.

Vector:

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}$$

In 3D,

$$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\vec{i} + y\vec{j} + z\vec{k}$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors along the x-, y- and z-axes:

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Magnitude of a position vector (distance from origin) or norm:

$$a = |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^3 + \cdots + a_n}$$

Angle between x-axis and the vector (2D):

$$\tan\theta = \frac{y}{x}$$

Point: $\mathrm{A}(a\cos\theta, a\sin\theta)$ where

$$\cos\theta = \frac{x}{a}, \quad \sin\theta = \frac{y}{a}$$

Transpose:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}^T = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix}$$

Addition (commutative):

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

Multiplication by a scalar:

$$k\vec{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \\ \vdots \\ ka_n \end{pmatrix}$$

Parallel vectors are scalar multiples of each other. Antiparallel if $k < 0$.

Zero vector:

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}$$

Dot or scalar product

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \sum a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n \\ &= |\vec{a}|\ |\vec{b}| \cos\theta \end{aligned}$$

Angle between two vectors:

$$\theta = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\ |\vec{b}|}\right)$$

Vectors are perpendicular if $\vec{a} \cdot \vec{b} = 0$ ($\theta = 90°$).

Component of $\vec{a}$ in the direction of $\vec{b}$ is

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

Cross or vector product (3D)

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Cross product is orthogonal to both vectors.

Magnitude:

$$|\vec{a} \times \vec{b}| = |\vec{a}|\ |\vec{b}| \sin\theta$$

which is the area of a parallelogram formed by $\vec{a}$ and $\vec{b}$.

Component of $\vec{a}$ perpendicular to $\vec{b}$ is

$$\frac{|\vec{a} \times \vec{b}|}{|\vec{b}|}$$

Vector equation of a straight line:

$$\begin{aligned} \vec{r} &= \vec{a} + \lambda\vec{b} \\ \vec{r} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \end{aligned}$$

where $\vec{a}$ is the position vector of a point on the line and $\vec{b}$ is a vector parallel to the line and $-\infty < \lambda < \infty$.

Equation of motion: $\vec{r} = \vec{r_0} + \vec{v}t$

Speed is $|\vec{v}|$

For bearing α and speed s, equation of motion:

$$\vec{r} = \vec{r_0} + st \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}$$

15.1 Translation vectors

$$\overrightarrow{AB} = \vec{b} - \vec{a} = -\overrightarrow{BA}$$

for position vectors $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$

Triangle law of vector addition:

$$\begin{aligned} \overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC} \\ \overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} \\ &\dots \end{aligned}$$

A non-linear transformation:

$$\vec{v'} = \vec{v} + \vec{a}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \end{pmatrix}$$

$$(x, y) \to (x+a, y+b)$$

16 Matrices

Dimension or order: $m \times n$ matrix has m rows and n columns: a_{ij}: Value in ith row and jth column:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Round $(\dots)$ or square $[\dots]$ braces are used interchangeably.

- 2×1: column matrix or column vector, or simply, vector (in 2D), e.g., $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- 1×2: row matrix or row vector (in 2D), e.g., $\begin{pmatrix} 1 & 2 \end{pmatrix}$

Addition

Commutative:

$$\begin{aligned} \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} \end{aligned}$$

Associative:

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

Scalar multiplication

$$k\mathbf{A} = k \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}$$

Matrix multiplication: not commutative

$$\begin{aligned}\mathbf{C} = \mathbf{A}\times\mathbf{B} = \mathbf{AB}\\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}\end{aligned}$$

If $\mathbf{A}$ has dimensions $m \times n$ and $\mathbf{B}$ has dimensions $p \times q$ then the product $\mathbf{AB}$ is defined only if the number of columns in $\mathbf{A}$ is equal to the number of rows in $\mathbf{B}$: $n = p$.

The dimensions of the product are $m \times q$.

Associative: $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

Multiplicative identity:

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Determinant and inverse of a square matrix

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Singular or degenerate matrix: $\det \mathbf{A} = 0$. The matrix has no inverse.

Determinant of 2×2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Adjugate:

$$\operatorname{adj} \mathbf{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse:

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \operatorname{adj} \mathbf{A} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Determinant of 3×3 matrix

$$\begin{aligned}\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\\ = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}\end{aligned}$$

Eigenvectors and eigenvalues

$$\mathbf{A}\vec{x} = \lambda\vec{x}$$

where $\mathbf{A}$ is a square matrix, $\vec{x}$ is an eigenvector and λ is an eigenvalue (a scalar, possibly complex).

Find eigenvalues from polynomial:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Find eigenvectors from simultaneous equations:

$$(\mathbf{A} - \lambda_1\mathbf{I})\vec{x_1} = 0$$

16.1 Linear transformations

A transformation f is linear iff:

$$f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$$
$$f(k\vec{x}) = kf(\vec{x})$$

i.e., addition and scalar multiplication are preserved.

$$\vec{v'} = \mathbf{A}\vec{v}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$(x, y) \mapsto (ax + by, cx + dy)$$

For a 2D shape:

$$\text{Area of image} = |\det \mathbf{A}| \times \text{Area of object}$$

Linear transformation		Matrix
Identity		
No change	$(x,y) \mapsto (x,y)$	$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflections		
About x-axis	$(x,y) \mapsto (x,-y)$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
About y-axis	$(x,y) \mapsto (-x,y)$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
About $x = a$	$(2a - x, y)$	
About $y = a$	$(x, 2a - y)$	
About $y = (\tan\theta)x$		$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
Central about origin (equivalent to rotation of 180°)	$(x,y) \mapsto (-x,-y)$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
Central about (a,b)	$(x,y) \mapsto (-x+2a, -y+2b)$	
Rotations		
Anticlockwise about $(0,0)$ $\theta > 0$		$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
Clockwise about $(0,0)$ $\theta > 0$		$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

For rotations and for positive θ anticlockwise:

$90°$	$(x, y) \mapsto (-y, x)$
$180°$	$(x, y) \mapsto (-x, -y)$
$270°$	$(x, y) \mapsto (y, -x)$
$360° = 0°$	$(x, y) \mapsto (x, y)$

Scaling

$$\vec{v'} = k\vec{v}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = k \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$$

$$(x, y) \mapsto (kx, ky)$$

- $k > 1$: enlargement or dilation
- $k = 1$: no change
- $0 \leq k < 1$: squeezing
- $k = 0$: reduction to the origin, $(0, 0)$
- $k < 0$: reflection in the origin plus scaling

Horizontal stretch parallel to x-axis with a scale factor of k	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
Vertical stretch parallel to y-axis with a scale factor of k	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
Enlargement, with a scale factor of k, centre $(0, 0)$	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

17 Graphs

17.1 Glossary

vertex A point or node on a graph

edge A line connecting two vertices

loop An edge that starts and ends at the same vertex; each loop of a directed graph counts as 1 and each loop of an undirected graph counts as 2

order The number of vertices in a graph

size The number of edges in a graph

degree of a vertex The number of edges that are incident with the vertex

in-degree of a vertex The number of adjacent edges that have a direction into the vertex

out-degree of a vertex The number of adjacent edges that have a direction out of the vertex

adjacent vertices Two vertices with a common edge

adjacent edges Two edges with a common vertex

leading diagonal The diagonal from top left to bottom right of an adjacency matrix

weight A number assigned to an edge that represents some property of that edge, for example distance, time or cost

directed edge An edge that has a specific direction associated with it, for example, a one-way street

transition matrix A matrix that that contains the probabilities of travelling from a vertex in a graph to any other vertex in the graph.

17.2 Types of graph

planar graph A graph that can be drawn such that no edges cross each other

simple graph A graph that does not contain any loops or vertices with multiple edges; all cells are 0 or 1 and the leading diagonal has only 0's

multigraph A graph that has vertices with multiple edges or vertices with loops

multiple edges (multi-edge) When two vertices are connected by more than one edge

weighted graph A graph that contains edges that have been given weights

complete graph A simple graph, K_n, such that each of the n vertices is adjacent to every other vertex; complete graphs with the same number of vertices are isomorphic to each other

directed graph (digraph) A graph that contains directed edges

subgraph A graph whose components are contained within another graph

tree A connected graph that contains no cycles

minimum spanning tree The subgraph of an undirected weighted graph that is of lowest weight, connects all the vertices in the graph, and does not contain any cycles

connected graph A graph that contains a path from every vertex to every other vertex; a directed graph is connected if a walk can be constructed in at least one direction between any two vertices; it has an even number of vertices with odd degree

strongly connected graph A directed graph that contains a path from every vertex to every other vertex

weakly connected graph A directed graph that is connected but not strongly connected

disconnected graph A graph that does not contain a path from every vertex to every other vertex

Eulerian graph A graph that contains an Eulerian circuit

Hamiltonian graph A graph that contains at least one Hamiltonian cycle

isomorphic graphs If two graphs have the same number of vertices and the vertices are connected in the same way, the two graphs are isomorphic to each other, that is they can be considered to be identical

17.3 Walks, paths, etc.

walk A sequence of connected edges in a graph

random walk A walk on a graph that is created by randomly selecting the next vertex to visit

path A walk on a graph that has no repeated vertices

trail A walk on a graph that has no repeated edges

circuit A trail on a graph that begins and ends at the same vertex, also known as a closed trail

cycle A walk that begins and ends at the same vertex and has no other repeated vertices; all vertices must have even degree

length of a walk In an unweighted graph, the number of edges in the walk; in a weighted graph, the sum of the weights of the edges in the walk

Eulerian trail A trail that contains all edges within a graph. For an undirected graph, an Eulerian trail must begin at a vertex with an odd degree and end at the only other vertex with an odd degree. For a directed graph, an Eulerian trail begins at a vertex with out-degree one more than its in-degree and ends at a vertex with in-degree one more than its out-degree; for all other vertices, the in-degree must equal the out-degree.

Eulerian circuit A closed trail that contains all edges. An undirected graph has an Eulerian circuit if and only if every vertex has even degree. A directed graph has an Eulerian circuit if and only if the in-degree equals the out-degree for each vertex.

Hamiltonian path A path that contains all vertices

Hamiltonian cycle A cycle that includes every vertex with no vertex included more than once
Number of different Hamiltonian cycles within a complete graph:

$$\frac{1}{2}(n-1)!$$

17.4 Adjacency matrices

An adjacency matrix gives the number of edges between two vertices:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Value in ith row and jth column is a_{ij}.

For an unweighted graph, a_{ij} is the number of edges between i and j.

For a directed graph, a_{ij} is the weight to go from i to j.

The number of walks of length n from vertex i to vertex j is the entry in the ith row and the jth column of $\mathbf{A}^n$.

The number of walks of length r or less between any two vertices is given by $\mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^r$.

17.5 Transition matrices

a_{ij} is the probability of going from j to i (opposite to an adjacency matrix). Values in each column must sum to 1:

$$\sum_j a_{ij} = 1$$

Steady-state probabilities are given by a vector found by solving $\mathbf{T}\vec{x} = \vec{x}$ plus $|\vec{x}| = 1$.

An absorbing state is one that cannot transition to another.

17.6 Kruskal's algorithm

This finds the minimum spanning tree of a graph by selecting the edges with the lowest weights without creating a cycle.

1. Find the edge of least weight anywhere in the graph. If there are two or more edges with the same weight, any may be chosen.
2. Add the edge of least weight that has not already been selected and does not form a cycle with the previously selected edges.
3. Repeat step 2 until all the vertices are connected.

17.7 Prim's algorithm

This finds the minimum spanning tree of a graph by selecting the vertex that is closest to any vertex that has already been selected.

1. Select any vertex and add the edge of least weight adjacent to it.
2. Add the edge of least weight that is incident to the tree formed in first step and does not connect to a vertex already in the tree.
3. Repeat step 2 until all the vertices have been connected.

17.8 Chinese postman problem

To work out the circuit with the shortest route by finding an Eulerian circuit. An Eulerian circuit exists if and only if every vertex has even degree. If an Eulerian circuit does not exist, edges are duplicated as follows so that the graph becomes Eulerian:

1. Determine the degree of each vertex.
2. For each odd vertex, list each combination of pairs.
3. Find the shortest path between each pair.
4. For the combination of pairs with the lowest overall length, duplicate the edges in the paths.

17.9 Travelling salesman problem

The aim of this problem is to find the shortest Hamiltonian cycle, i.e., start at a vertex, visit every vertex and then return to the starting vertex.

The nearest neighbour algorithm

This produces a cycle that is not necessarily a Hamiltonian cycle, but the overall weight is an upper bound for the problem:

1. Pick a vertex of the graph as a starting point.
2. Go from the current vertex to the nearest adjacent vertex that has not already been visited.
3. Repeat step 2 until all vertices have been reached.
4. Complete the cycle by travelling back to the starting vertex.
5. Repeat using a different initial vertex. The best upper bound is the one with the lowest weight.

Deleted vertex algorithm

This gives a lower bound to the travelling salesman problem:

1. Choose a vertex and remove it and all the edges incident to it from the graph.
2. Find the minimum spanning tree for the remaining subgraph.
3. The lower bound is the weight of the minimum spanning tree plus the combined weight of the two edges of least weight removed in step 1.
4. Repeat using a different initial vertex. The best lower bound is the one with the largest weight.

18 Interest

Principal, amount initially invested: P

Interest, as a percentage R (e.g., 5%) or a rate r (e.g., 0.05). Positive for investments; negative for loans. $r = R/100$.

Total interest paid, negative for loans: I

Number of periods (typically, years or months): n

Final amount:

$$\mathrm{A} = \mathrm{P} + \mathrm{I}$$

18.1 Simple interest

Interest is paid only on amount initially invested or borrowed. Simple interest is an arithmetic series:

$$\begin{aligned} \mathrm{I} &= P + Pr + Pr + \ldots \\ &= P \times r \times n \\ &= P \times \frac{R}{100} \times n \end{aligned}$$

$$\begin{aligned} A &= P + (P \times r \times n) \\ &= P + \left(P \times \frac{R}{100} \times n\right) \end{aligned}$$

18.2 Compound interest

Interest is paid on interest. Compound interest is a geometric series.

Paid per period:

$$A = P \times (1 + r)^n$$

Amount to invest:

$$P = \frac{A}{(1 + r)^n}$$

Paid for a number of periods in a year:

$$A = P \times \left(1 + \frac{r}{k}\right)^{kn}$$

where k is the number of periods in 1 year and n is the number of years.

Inverse functions:

$$n = \frac{1}{k} \log_{1+(r/k)} \left(\frac{A}{P}\right)$$

$$r = k\left(\left(\frac{A}{P}\right)^{1/kn} - 1\right)$$

Added continuously:

$$A = P \times \mathrm{e}^{100r/n}$$

Repayment amount:

$$a = P\frac{r(1 + r)^n}{(1 + r)^n - 1}$$

Amount a is paid for n periods with interest rate r added each period.

Depreciation reducing balance:

$$A = P \times (1 - r)^n$$

19 Logic

Symbol	Name	Common language
$\neg$	Negation	not
$\wedge$	Conjunction	and
$\vee$	Disjunction	either ... or (or both)
$\veebar$	Exclusive disjunction	either ... or ... but not both
$\Rightarrow$	Implication	implies
$\Leftrightarrow$	Equivalence	is equivalent to

19.1 Truth tables

AND	false	true
false	false	false
true	false	true

AND	0	1
0	0	0
1	0	1

OR	false	true
false	false	true
true	true	true

OR	0	1
0	0	1
1	1	1

NAND	false	true
false	true	true
true	true	false

NAND	0	1
0	1	1
1	1	0

XOR	false	true
false	false	true
true	true	false

XOR	0	1
0	0	1
1	1	0

20 Temperature

Fahrenheit to centigrade:

$$C^\circ = (F^\circ - 32) \times \frac{5}{9} = \frac{(F^\circ - 32)}{1.8}$$

Centigrade to Fahrenheit:

$$F^\circ = 32 + C^\circ \times \frac{9}{5} = 32 + C^\circ \times 1.8$$

Absolute zero: $0\,\mathrm{K} = -273.15°\mathrm{C}$

21 Glossary

2D Two dimensions, two-dimensional; relating to a plane

3D Three dimensions, three-dimensional; relating to a *solid*

abscissa Value of a coordinate on the horizontal or x-axis. It is used for the independent variable. cf. *ordinate*

acute angle $0 < \theta < 90°$

arc Part of the circumference of a circle or another curve. For a circle, two points divide the circumference into the minor arc and the major arc.

bisect Cut into two equal parts. Angles and lines can be bisected.

bisector A line that divides an angle or another line into two equal parts. cf. *perpendicular bisector*

bar chart Used for discrete variables (e.g., number of cars of different colours). The height of a bar is proportional to the number. cf. *histogram*

cardinal number A counting number or natural number: $0, 1, 2, 3, \ldots$ cf. *ordinal number*

Cartesian product of two sets $A \times B = \{(a, b) \mid a \in A, b \in B\}$: The set of all ordered pairs where the first member of the pair is from A and the second from B.

central angle In a circle, an angle whose vertex is the centre of the circle and whose end points are on the circumference of the circle

chord A straight line segment connecting two points on the circumference of a circle or another curve.

circle A two-dimensional *curve* that is always *equidistant* from a point called the center.

circumcentre The centre of a *circumcircle.*

circumcircle A circle outside a polygon that passes through all the vertices of the polygon.

circumscribed Of a shape, enclosing or surrounding another shape; cf. *inscribed.*

circumscribed circle A *circumcircle.*

circumscribed polygon A polygon outside a circle such that each side of the polygon is a tangent to the circle.

coefficient In science, a coefficient is a constant that represent an important physical property. In maths, a coefficient is often a constant, such as the coefficients of x in a polynomial. For example, in $ax^2 + bc + c$, a, b and c are coefficients. x is a *variable.* cf. *expression*

collinear If two or more points lie on the same straight line, they are collinear. Since a straight line can always be drawn between two points, two points are always collinear. cf. *coplanar*

complex number A number written like $x + \mathrm{i}\, y$, where $\mathrm{i} = \sqrt{-1}$. Represented as $\mathbb{C}$. cf. *real number*

cone A pyramid with a circular base.

continuous A function $f(x)$ is continuous if it has no breaks or jumps.

converse Consider a theorem such as: 'If A then B.' The converse is: 'If B then A.' The converse is not always true. An example where it is true: 'If a triangle is equilateral, then all its angles are 60°.' The converse is: 'If all the angles in a triangle are 60°, then it is equilateral.' cf. *iff*

convex polygon A polygon with all its interior angles less than 180°.

coplanar If two or more lines, polygons or other 2D shapes lie in the same plane, they are coplanar. cf. *collinear*

cosine A trigonometry ratio: cos = adjacent / hypotenuse.

counter diagonal The diagonal in a matrix from top right a_{1n} to bottom left a_{m1}. cf. *main diagonal.*

counterexample An example that contradicts an assumption. Assumption: All primes are odd. Counterexample: 2, which is prime and even. Therefore, not all primes are odd.

cube 1. A 3D shape with six square faces that all meet at 90°. One of the regular polyhedrons. 2. A cubic number (e.g., $3^3 = 81$).

cuboid A 3D shape with six rectangular faces that all meet at 90°. cf. *cube*, *parallelepiped*

curve 1. A non-straight part of an object, such as an *arc* of a *circle.* 2. A line on a *graph.* representing a relation between two variables.

curvilinear A curvilinear line is not straight. It has curves.

cylinder A *prism* with a circular cross section.

denominator The number or expression written below the line in a fraction, such as 2 in $\frac{1}{2}$; cf. *numerator.*

dense matrix A matrix in which most of the elements are not zero; cf. *sparse matrix.*

dependent variable On a graph, the variable plotted along the ordinate or y-axis. In $y = f(x)$, y is the dependent variable. The dependent variable is often measured or calculated for a range of values of the independent variable. cf. *variable*, *independent variable*, *ordinate*

diagonal A line joining the opposite vertices of a polygon or matrix. cf. *main diagonal*, *counter diagonal*

differential equation An equation with a derivative, e.g.,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \text{ or } \frac{\partial y}{\partial t} = x^2$$

edge A line segment connecting two vertices. cf. *vertex.* For example, the sides of a triangle or other polygon are edges.

equality or equation Expresses that two things are equal: $y = x$. cf. *inequality*, *identity*

equidistant At the same distance.

equilateral triangle A triangle that has three equal sides. All angles are 60°.

expression Anything expressed mathematically, such as a term, polynomial, surd, equation, etc.

extrapolation If we have a function $y = f(x)$ and some known values of both x and y, such as (x_1, y_1), (x_2, y_2), (x_3, y_3), $\ldots$, then in extrapolation, we estimate the value of y, say y', for some value of x, say x', where x' lies *outside* the range of the values of x. For example, if the population of a small town was 1000 in 2019, 1010 in 2020 and 1020 in 2021, we could extrapolate to 2022 by predicting that the town will have 1030 people in 2022. cf. *interpolation*

factor A positive integer that divides anther positive integer exactly, including the number itself and 1. For example, the factors of 6 are 1, 2, 3 and 6. Also used for a polynomial that divides another polynomial. cf. *greatest common divisor*

factorial $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$ for $n \in \mathbb{N}$. For example, $1! = 1$, $2! = 2 \times 1 = 2$, $3! = 3 \times 2! = 3 \times 2 \times 1 = 6$, etc. Finally, $0! = 1$, by convention.

finite Not infinite. A variable that is finite has a value such as 100. cf. *infinity*

formula An *equality or equation.* A formula is often used to calculate something, like the conversion from centigrade to Fahrenheit.

frustum A cone or pyramid whose tip has been removed by a cut through a plane that is parallel to its base.

gradient The slope of a line or curve, measured as the ratio of the change in y over the change in x: $\Delta y/\Delta x$ or $\mathrm{d}y/\mathrm{d}x$. cf. *tangent*

graph 1. A *curve* or curves plotted on a set of axes. 2. A set of points or vertices connected by edges.

greatest common divisor $\gcd(a, b)$: The largest integer that is a factor of both a and b. For example, $\gcd(8, 12) = 4$ and $\gcd(9, 12) = 3$. cf. *factor*

histogram A frequency distribution of a continuous variable (e.g., heights and ages). The height of a bar is proportional to the number or frequency. cf. *bar chart*

hypotenuse The longest side in a right-angled triangle; it is opposite the right angle.

identity Represented by $\equiv$. Like equals (=) but true for all values of the variables. For example, $y = x$ is true for $x = 1$ and $y = 1$ but not true for $x = 1$ and $y = 2$. In contrast, $\sin^2\theta + \cos^2\theta \equiv 1$ is true for all values of θ. cf. *equality or equation*

iff If and only if. A iff B, means that if A is true then B is true and if B is true then A is true. cf. *converse*

imaginary number A complex number; a number of the form $a + b\,\mathrm{i}$ where $i = \sqrt{-1}$.

independent variable On a graph, the variable plotted along the abscissa or x-axis. In $y = f(x)$, x is the independent variable. The independent variable is often scanned over a range of values and the dependent variable is then measured or calculated for each of these values. cf. *variable*, *dependent variable*, *abscissa*

inequality Expresses that two things are not equal: $y < x$ or $y > x$; cf. *equality or equation.*

infinitesimal An indefinitely small number. A number that is being made smaller and smaller until, in the limit, it is 0. This idea is used in calculus.

infinity A value greater than any possible number. We use the symbol ∞. For example, the natural numbers continue without ending, since whatever number we have reached we can always add on 1. Thus, we write: $1, 2, 3, \ldots, \infty$. cf. *finite*

inflection point See *point of inflection*

inscribed Of a shape, being enclosed by or inside another shape; cf. *circumscribed.*

inscribed circle A circle inside a polygon such that each side of the polygon is a tangent to the circle.

inscribed polygon A polygon inside a circle such that the vertices of the polygon lie on the circumference of the circle.

integer A whole number, represented by $\mathbb{Z}$: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

interpolation If we have a function $y = f(x)$ and some known values of both x and y, such as (x_1, y_1), (x_2, y_2), (x_3, y_3), ..., then in interpolation, we estimate the value of y, say y', for some value of x, say x', where x' is within the range of the values of x. cf. *extrapolation*

irrational number A number that cannot be written as a fraction or ratio, such as surds, π and e. cf. *rational number*

isosceles triangle A triangle that has two equal sides.

line 1. A straight line: the shortest distance between two points. 2. A line that is not straight, i.e., a *curve.* cf. Section 10.3

line segment A portion of a straight line between two points. It has a finite length.

locus A set of points that satisfy the same condition or rule, such as the points on a line or circle. The plural is 'loci'.

lowest common denominator The lowest common multiple of the denominators of several fractions.

lowest or least common multiple $\mathrm{lcm}(a, b)$: The smallest positive integer that is a multiple of a and b. For example, $\mathrm{lcm}(2, 3) = 6$. It is used when adding or subtracting fractions.

magnitude A numeric quantity expressing the size of something, such as the magnitude of a vector. cf. *order of magnitude*

main diagonal The diagonal in a matrix from top left a_{11} to bottom right a_{mn}. cf. *counter diagonal.*

measure In geometry, this refers to the magnitude of something, e.g., the size of an angle (in degrees or radians) or the length of a line segment (in appropriate units of length, such as the metre or centimetre).

metric 1. A number representing the size or magnitude of something. For example, IQ is a metric for intelligence. 2. A measure of the distance between two points.

natural number A counting number or a positive integer. Sometimes, a non-negative integer (i.e., 0 may or may not be included). Denoted by $\mathbb{N}$: $1, 2, 3, \ldots$

nominal data In statistics, data that cannot be ordered, e.g., colours, gender. cf. *ordinal data*

numerator The number or expression written above the line in a fraction, such as 1 in $\frac{1}{2}$; cf. *denominator*

obtuse angle $90° < \theta < 180°$

order of magnitude A rough estimate of the size of a number by rounding it to the nearest power of 10. Thus, 20 and 30 are both of the same order of magnitude (10) whereas 200 and 300 are an order larger (100). cf. *magnitude*

ordinal data In statistics, data that can be ordered because it is associated with a number, e.g., heights, weights. cf. *nominal data*

ordinal number A number denoting position or order: first, second, third, etc. cf. *cardinal number*

ordinate Value of a coordinate on the vertical or y-axis. It is used for the dependent variable. cf. *abscissa*

parabola The shape of a quadratic function.

parallel Two lines are parallel if they lie in the same plane (coplanar) and never meet. Informally, they are said to meet at infinity. cf. *perpendicular*, *skew lines*

parallelepiped A 3D shape or polyhedron with six faces that are all parallelograms. Opposite faces are similar and parallel.

parallelogram A plane figure with four sides (quadrilateral) where opposite sides are parallel. cf. *rhombus*

parameter A number that is used as an input to a function.

perfect square An integer that is the square of an integer, such as $1 = 1^2$, $4 = 2^2$ and $9 = 3^2$.

perpendicular Two lines are perpendicular if they intersect at a right angle. cf. *parallel*

perpendicular bisector A line that divides a line segment into two equal parts and is perpendicular to the segment.

plane figure A 2D figure or geometric shape that can line on a plane, such as a triangle or circle. cf. *solid*

point of inflection A point where a *curve* changes from concave (concave down) to convex (concave up) or vice versa. A horizontal point of inflection is a *stationary point*; also spelt 'inflexion'

polygon A closed plane figure bounded by straight lines, such as a triangle, rectangle, hexagon, etc.

polyhedron A *solid* formed by plane faces, such as a *cube*, pyramid, tetrahedron, etc.

polynomial An expression with only non-negative integral powers of x multiplied by a *coefficient*: $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4^x4 + \cdots + a_nx^n$

prime number A natural number greater than 1 that cannot be formed by multiplying together two natural numbers. Its only divisors are 1 and the number itself. The only even prime is 2. The first few primes are: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \ldots$

prism A *solid* with a uniform cross section, such as a *cylinder* or *cuboid*.

quadrant One of the four quarters of a plane separated by the x- and y-axes.

quadrilateral A plane figure with four sides, such as a parallelogram or rectangle

radian If an arc of a circle is the same length as the radius, then the angle subtended by the arc is 1 radian. One radian is just under 57.3°.

rational number A number that can be written as a fraction or ratio, represented by $\mathbb{Q}$, such as $1/2$, $-1/2$, $2/1 = 2$ and 0. cf. *irrational number*

real number Reals comprise integers, rational numbers, irrational numbers and transcendental numbers, represented by $\mathbb{R}$. cf. *complex number*

reciprocal Swapping the numerator and denominator in a fraction. For example, p/q and q/p are reciprocals, as are $1/5$ and 5

rectangle A plane figure with four sides (quadrilateral), where opposite sides are parallel (parallelogram) and all angles are 90°. cf. *square*

recurring decimal A decimal in which one or more digits occurs indefinitely. For example, $1/3 = 0.33333... = 0.\dot{3}$, $3/3 = 0.99999... = 0.\dot{9} = 1$ and $9/11 = 0.818181... = 0.\dot{8}\dot{1}$.

reflex angle $180° < \theta < 360°$

regular polygon A polygon with all sides the same length and all interior angles the same measure; e.g., equilateral triangles and squares.

regular solid A solid where each face is a regular polygon, also called a Platonic solid. There are five of them: tetrahedron (4 faces, each an equilateral triangle), cube (6 faces, each a square), octahedron (8 faces, each an equilateral triangle), dodecahedron (12 faces, each a regular pentagon) and an icosahedron (20 faces, each an equilateral triangle).

residual The difference between an observed or measured value and a value predicted by a model, such as a best-fit line.

right angle $\theta = 90°$

rhombus A *parallelogram* in which all the sides are the same length. cf. *square*

root 1. The solution of an equation, especially a quadratic equation. 2. A number raised to a power gives a specified number, such as a square root or cubic root.

scalene triangle A triangle in which all three sides (and thus angles) are unequal.

sector The area between two radii and the connecting arc of a circle.

segment The region between a chord of a circle and its associated arc. cf. *line segment*

sequence A succession of ordered terms $a_1, a_2, a_3, \ldots$, either infinite or finite; cf. *series*, *set*.

series The sum of a *sequence*: $\sum a_i$.

set A collection of items: $\{a_1, a_2, a_3, \ldots\}$. Unlike a *sequence*, a set is not ordered and cannot contain duplicates. Sets may be finite (e.g., $\{1, 2, 3\}$) or infinite (e.g., $\mathbb{R}$).

sine A trigonometry ratio: sin = opposite / hypotenuse.

skew lines In 3D, lines that are neither intersecting nor parallel.

solid A 3D figure such as a cylinder, cube, cuboid, parallelepiped, pyramid, sphere or tetrahedron. cf. *plane figure*, *regular solid*

sparse matrix A matrix in which most of the elements are zero. cf. *dense matrix*

square 1. A rectangle where all sides have the same length. 2. A square number (e.g., $3^2 = 9$).

stationary point A point on a curve where the tangent is horizontal, $\mathrm{d}y/\mathrm{d}x = 0$. A maximum, minimum or horizontal *point of inflection*. cf. *turning point*

subset A set is a subset of another of set if all its members are in the other set.

subtend Subtend means to form the central angle underneath an arc.

surd An expression with an irrational root, such as $\sqrt{3}$, $\sqrt{2} + 5$, etc.

tangent 1. A line that touches a curve. 2. A trigonometry ratio: tan = opposite / adjacent. cf. *gradient*

trapezium A quadrilateral where two sides are parallel but the other two are not.

triangle A plane figure with three sides.

turning point A local maximum or minimum on a graph.

units In geometry, 'units' (length), 'units^2' (area) or 'units^3' (volume) are often used instead of actual units (e.g., m, m^2 or m^3) because mathematicians are commonly more interested in the numbers than what they represent.

unit circle A circle with a radius of 1 unit.

variable A variable is something like x or y that stands for a number in a formula. For example, in $y = ax^2 + bc + c$, x and y are variables. The variables may be plotted on a graph as the abscissa and ordinate. a, b and c are coefficients (though they are sometimes referred to as variables). Depending on the context, a variable can be any sort of number, or a vector, a matrix and so on. The sort of value a variable can take is usually defined using a set, such as $x \in \mathbb{R}$. cf. *abscissa*, *ordinate*, *coefficient*, *dependent variable*, *independent variable*

vertex A point where two lines intersect. For example, a triangle has three vertices and a rectangle four. cf. *edge*.

wrt 'with respect to'. For example, $\mathrm{d}x/\mathrm{d}t$ is the rate of change of x wrt t.